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## **Spectral characterizations of graphs**

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*Publication date:*  
2015

*Document Version*  
Publisher's PDF, also known as Version of record

[Link to publication in Tilburg University Research Portal](#)

*Citation for published version (APA):*

Abiad Monge, A. (2015). *Spectral characterizations of graphs*. [Doctoral Thesis, Tilburg University]. CentER, Center for Economic Research.

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# Spectral Characterizations of Graphs

PROEFSCHRIFT

Proefschrift ter verkrijging van de graad van doctor aan Tilburg University  
op gezag van de rector magnificus, prof.dr. E.H.L. Aarts, in het openbaar  
te verdedigen ten overstaan van een door het college voor promoties  
aangewezen commissie in de aula van de Universiteit op

vrijdag 19 juni 2015 om 10.15 uur

door

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geboren op 28 januari 1985 te Barcelona, Spanje.

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# Acknowledgements

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To my supervisor Willem Haemers from whom I have learnt so much, and whose patience, wisdom, and unconditional support have been invaluable to me. His advice, not only professional, always inspired and motivated me, and I cannot think of a better supervisor to have. To my second supervisor Edwin van Dam; for his valuable comments, for his friendliness and for his advice, but most of all, for being always available for questions and sudden drop-ins into his office. To Miquel Angel Fiol, who will always be my catalan supervisor; my deepest thanks for giving me the chance to switch my professional path and discover the beauty of mathematics. To the members of my committee for their valuable feedback. To NWO and Tilburg University for their financial support which enabled me to complete this thesis and to attend several conferences. To Jack Koolen; his hospitality, patience and stimulating discussions made my visit to China an unforgettable experience. To my coauthors; working together has been fun and productive. To my friends; for being an amazing company and for showing me that I can count on you. And finally, to my family for their constant love and support.

Without the guidance and friendship of the people mentioned in Figure 1, this thesis would not have been completed. I extend my deepest gratitude to all of you in my favourite graph: the acknowledgement graph.

Aida Abiad  
Tilburg, May 2015

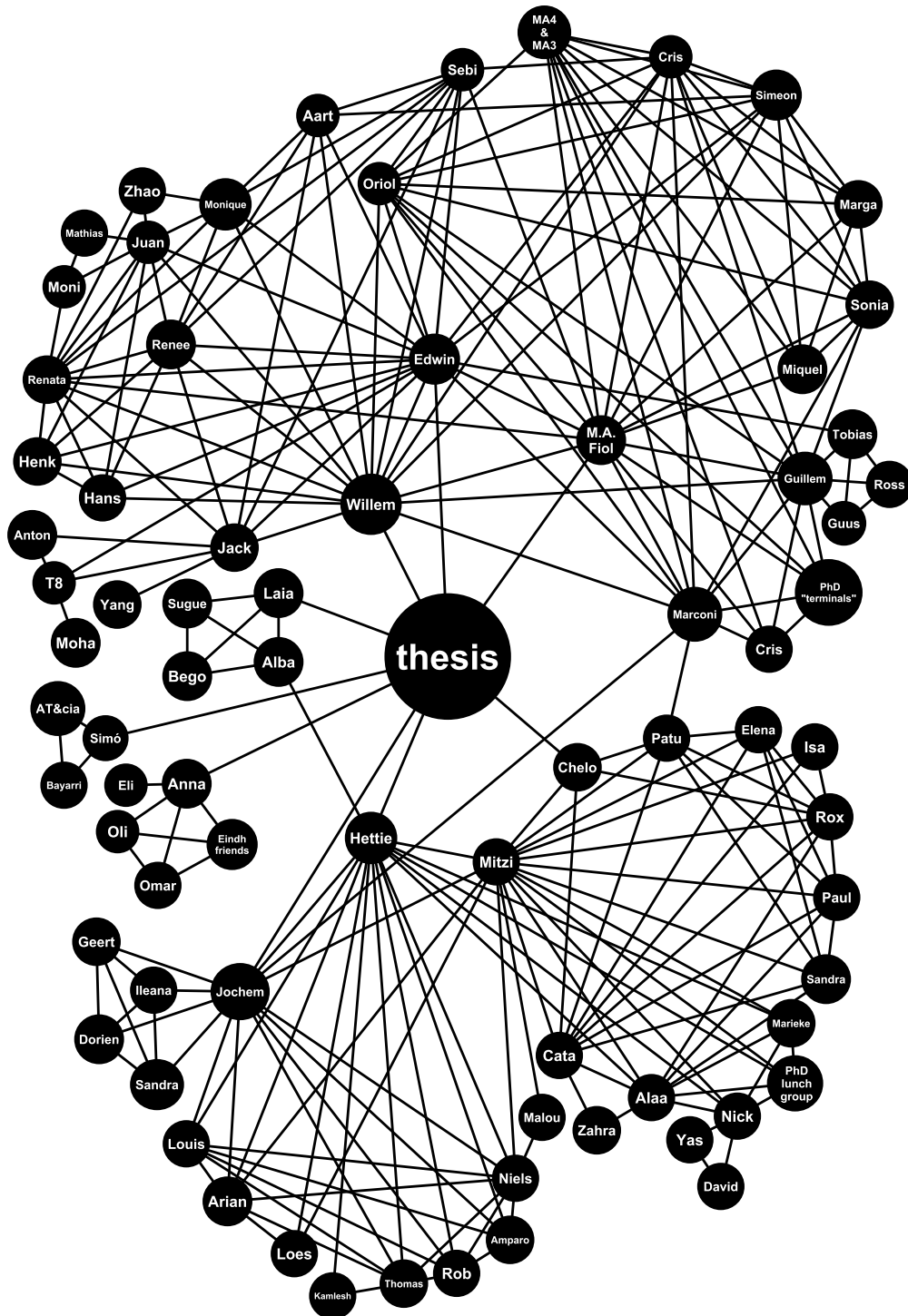


Figure 1 The acknowledgement graph.

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# Introduction

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Spectral graph theory studies the relation between structural properties of the graph and the eigenvalues of associated matrices. Graphs are often studied by their adjacency matrix, a square zero-one matrix whose rows and columns are both indexed in the same order by the vertices of the graph, with a 1 in a given position if and only if the corresponding vertices are adjacent. In this thesis we will also consider other types of matrices (generalized adjacency matrix and Laplacian matrix). If we do not specify the matrix, we assume we are dealing with the adjacency matrix.

The spectrum of a finite graph is by definition the spectrum of the adjacency matrix, that is, its set of eigenvalues together with their multiplicities. The spectrum contains a lot of information of the graph, but in general it does not determine the graph (up to isomorphism). So a central question is:

*Given the spectrum of a graph, what can be said about its structure?*

For example, we can see from the spectrum whether the graph is regular, or bipartite. Spectral graph theory looks at answering questions of this type. Sometimes the eigenvalues uniquely determine the graph. If that is the case we say that the graph is determined by the spectrum (DS for short). In recent years the problem of determining whether the spectrum determines the graph has attracted much interest. Wang and Xu [69, 70, 68] defined a large family of graphs (which may have positive density among all graphs, as suggested by some numerical experiments) and showed that every graph in this family is determined by its spectrum and the spectrum of its complement. On the other hand, for graphs with a very special structure, such as trees and strongly regular graphs, it has been proved that they are almost never determined by the spectrum (see [49], [61]). For many graphs,

it has been established whether they are determined by the spectrum or not. However, for many other interesting graphs the problem is still open. See [60] and [61] for a general survey of this problem.

For example, several distance-regular graphs are proved to be determined by their spectrum (for example, the Odd graphs), and for even more families of graphs nonisomorphic cospectral graphs have been constructed [59], [63], [41] (for example, the Johnson graphs  $J(n, k)$  are not determined by the spectrum if  $n/2 \geq k \geq 3$ ). Nevertheless, for many important families of graphs the problem is still unsolved.

Two graphs with the same spectrum for some type of matrix are called cospectral with respect to the corresponding matrix. Cospectral graphs help us understand “weaknesses” in identifying structures only using the spectrum. Consider the two graphs shown in Figure 1.1. It is easily checked that the corresponding adjacency matrices have spectrum

$$\{2^1, 0^3, -2^1\},$$

where the exponents indicate multiplicities. This is the first example of nonisomorphic cospectral graphs found by Collatz and Sinogowitz [19] in 1957. For graphs on less than five vertices, no pair with cospectral adjacency matrix exists, so any graph with less than five vertices is determined by its spectrum.



**Figure 1.1** Two cospectral graphs on 5 vertices.

If a graph is not determined by the spectrum, this can be proved by constructing a nonisomorphic cospectral mate. Several tools for constructing cospectral graphs are known to exist (see [60]); the most important one is the switching method of Godsil and McKay [33]. Godsil-McKay switching is an operation on a graph that does not change the spectrum of the adjacency

matrix (though it was invented to make cospectral graphs with respect to the adjacency matrix, the idea also works for the Laplacian matrix). So Godsil-McKay switching provides a tool for constructing cospectral mates for certain graphs. Constructing cospectral graphs is not only important for disproving that a graph is determined by its spectrum. In several cases such a graph can be important in its own right. Good examples are the twisted Grassmann graphs, found by Van Dam and Koolen [64], which form a new family of distance-regular graphs and which are cospectral with Grassmann graphs.

Results by Wang and Xu [69] are the inspiration for Chapter 3, where we present a new method to construct families of cospectral graphs that generalizes Godsil-McKay switching. In this chapter we will make use of regular (constant row sum) orthogonal matrices of level 2. We say that a matrix  $Q$  has level  $l$  if  $l$  is the smallest positive integer such that  $lQ$  is an integral matrix. Since  $A$  and  $A'$  are symmetric,  $G$  and  $G'$  are cospectral precisely when  $A$  and  $A'$  are similar, that is, there exists an orthogonal matrix  $Q$  such that  $A' = Q^T A Q$ . If  $Q$  is a permutation matrix (i.e.  $Q$  is regular of level 1) then  $G$  and  $G'$  are isomorphic. So the next natural step is to study the case when  $G$  is nonisomorphic with  $G'$ . If  $G$  and  $G'$  are nonisomorphic, and there exist a regular orthogonal matrix  $Q$  of level 2 such that  $A' = Q^T A Q$ , we call  $G$  and  $G'$  semi-isomorphic. Semi-isomorphic graphs are  $\mathbb{R}$ -cospectral, which means that the matrices  $xI + yJ + zA$  and  $xI + yJ + zA'$  have the same spectrum for every  $x, y, z \in \mathbb{R}$ ,  $z \neq 0$ , where  $J$  and  $I$  are the all-one matrix and the identity matrix, respectively. Johnson and Newman [45] show that being  $\mathbb{R}$ -cospectral is equivalent to being cospectral with cospectral complements. It has been conjectured by Van Dam and Haemers that almost every graph is determined by its spectrum [60], or equivalently, that the proportion of graphs on  $n$  vertices that are determined by their spectrum goes to 1 as  $n \rightarrow 1$ . A weaker version states that almost every graph is determined by its spectrum together with that of its complement. Both conjectures are still open, but Wang and Xu [71] have a number of results that support them. They prove that for almost no graph there exists a graph semi-isomorphic with it, and in addition they provide experimental evidence showing that a positive fraction of all pairs of nonisomorphic  $\mathbb{R}$ -cospectral graphs, are in fact semi-isomorphic. This makes it interesting to investigate the concept of semi-isomorphism. By using the classification of regular orthogonal matrices of level 2 [69], we work out the requirements for this switching operation to work in case  $Q$  has one nontrivial indecomposable block of size 4, 6, 7, or 8. Size 4 corresponds to Godsil-McKay switching of level 2. The other

cases provide new methods for constructing  $\mathbb{R}$ -cospectral graphs. For graphs with eight vertices all of these constructions are carried out. As a result we find that, out of the 1166 graphs on eight vertices that are  $\mathbb{R}$ -cospectral to another graph, only 44 are not semi-isomorphic to another graph.

For Godsil-McKay switching to work the graph needs a special structure, called a Godsil-McKay switching partition. This switching partition of the vertices of a graph makes it possible to switch some of the edges such that the spectrum of the adjacency matrix does not change. However, the presence of this structure does not imply that the graph is not determined by its spectrum; it may be that after switching the graph is isomorphic with the original one. In Chapter 4 we investigate this phenomenon. We obtain some elementary necessary conditions for isomorphism after switching and show how they can be used to guarantee nonisomorphism after switching for some graph products.

Finding switching partitions that make the Godsil-McKay switching work (the so-called *Godsil-McKay switching sets*) in a given family of graphs is a nontrivial problem that has only been solved in some special cases, like for the Johnson graphs  $J(n, k)$  with  $n/2 \geq k \geq 3$  [63] and some Kneser graphs  $K(n, k)$  [40], which are both families of graphs belonging to the Johnson association scheme. Some graphs in the Johnson scheme are determined by its spectrum, like  $K(2k + 1, k)$  [44] (also known as Odd graphs, whose vertices represent the  $k$ -element subsets of a  $(2k + 1)$ -element set, where two vertices are adjacent if and only if their corresponding subsets are disjoint) and  $J(n, 2)$  for  $n \neq 8$  (see for example [66]). But for most graphs in the Johnson association scheme it is not known if such Godsil-McKay switching set exists. This provided the initial motivation for Chapter 5. It is well-known that if a graph  $G'$  has the same spectrum as a strongly regular graph  $G$ , then  $G'$  is also strongly regular with the same parameters as  $G$  (see for example [14]). Therefore Godsil-McKay switching also provides a tool to construct new strongly regular graphs from known ones. However, again there is no guarantee that the switched graph is nonisomorphic with the original graph. The elementary necessary conditions for isomorphism after switching mentioned earlier do not apply here, since the graphs are strongly regular and have a lot of structure. Therefore, in Chapter 5 we use the 2-rank of the graph to prove nonisomorphism after switching. By the 2-rank of the graph we mean the rank of the adjacency matrix over the finite field  $\mathbb{F}_2$ . In particular, we apply Godsil-McKay switching to an important family of strongly regular graphs: the symplectic graphs over  $\mathbb{F}_2$ . We prove that the 2-rank of the graph increases after switching. This shows that

the switched graph is a new strongly regular graph with parameters  $(2^{2\nu} - 1, 2^{2\nu-1}, 2^{2\nu-2}, 2^{2\nu-2})$  and 2-rank  $2\nu + 2$  when  $\nu \geq 3$ . For the symplectic graph on 63 vertices we also investigate repeated switching by computer and find many new strongly regular graphs with the above parameters for  $\nu = 3$  with various 2-ranks. Using these results and a recursive construction method for the symplectic graphs from Hadamard matrices, we obtain several graphs with the above parameters, but different 2-ranks for every  $\nu \geq 3$ .

In Chapter 6, we deal with distance-regular graphs. Distance-regular graphs are a key concept in algebraic graph theory. They have important connections with other branches of mathematics, such as incidence geometry, coding theory, group theory, design theory, as well as with other areas of graph theory. As stated in the preface of the book by Brouwer, Cohen and Neumaier [12], this is because “most finite objects bearing enough regularity are closely related to certain distance-regular graphs”. A distance-regular graph with diameter  $d$  has  $d + 1$  distinct eigenvalues and its spectrum can be obtained from the intersection array. Conversely, the spectrum of a distance-regular graph determines the intersection array [59]. However, in general the spectrum of a graph does not tell you whether it is distance-regular or not. So in the theory of distance-regular graphs an important question is:

*Can we see from the spectrum of a graph whether it is distance-regular?*

For many distance-regular graphs this is known to be the case. In Chapter 6, we give a new contribution to this question. By generalizing some results of Van Dam and Haemers [59, 60], among others, we prove distance-regularity using, in addition to the spectrum, some metric parameters of  $G$ . In particular, we present some results assuring that a graph  $G$  is distance-regular without requiring, as it is common in this area of research, that  $G$  is cospectral with a distance-regular graph satisfying some combinatorial conditions. Among others, we show distance-regularity for graphs with large girth or odd-girth using the preintersection numbers.

Finally, in Chapter 7, we make use of a completely different spectral technique which is related to partitioned matrices: eigenvalue interlacing. This tool gives information about substructures. In this chapter we deal with the Laplacian matrix. The Laplacian matrix of a graph is the matrix  $L = D - A$ , where  $A$  is the adjacency matrix and  $D$  is the diagonal matrix of vertex degrees. So in this chapter, we apply eigenvalue interlacing to obtain lower and upper bounds for the sums of Laplacian eigenvalues of graphs, and characterize the case of equality. This leads to generalizations of, and variations

on theorems by Grone [35], and Grone and Merris [36]. As a consequence we obtain inequalities involving bounds for some well-known parameters of a graph, such as edge-connectivity, and the isoperimetric number.

Most results described in this thesis have been already published. Chapter 3 is mainly based on [5], Chapter 4 on [1], Chapter 5 on [6], Chapter 6 on [7] and Chapter 7 on [4].

## Publications and Preprints

The content of this thesis is based on the following publications and preprints:

### Chapter 3:

[5] A. Abiad and W.H. Haemers, Cospectral graphs and regular orthogonal matrices of level 2, *Electron. J. Comb.*, 19(3):Research Paper 13, 2012.

### Chapter 4:

[1] A. Abiad, A.E. Brouwer, and W.H. Haemers, Godsil-McKay switching and isomorphism, *Electron. J. Linear Algebra*, to appear.  
arXiv: [arxiv.org/abs/1406.4170](http://arxiv.org/abs/1406.4170)

### Chapter 5:

[6] A. Abiad and W.H. Haemers, Switched symplectic graphs and their 2-ranks, *preprint*, 2014.  
arXiv: [arxiv-web3.library.cornell.edu/abs/1412.2945v1](http://arxiv-web3.library.cornell.edu/abs/1412.2945v1)

### Chapter 6:

[7] A. Abiad, E.R. van Dam, and M.A. Fiol, Some Spectral and Quasi-Spectral Characterizations of Distance-Regular Graphs, *preprint*, 2014.  
arXiv: [arxiv.org/pdf/1404.3973v2.pdf](http://arxiv.org/pdf/1404.3973v2.pdf)

### Chapter 7:

[4] A. Abiad, M.A. Fiol, W.H. Haemers, and G. Perarnau, An Interlacing Approach for Bounding the Sum of Laplacian Eigenvalues of Graphs, *Linear Algebra Appl.*, 34:11-21, 2014.





# 2

## Background

---

This chapter presents some basic results on graph theory and graph spectra. For details and an overview of the results on spectra of graphs, we refer to the book by Brouwer and Haemers [14].

### 2.1 Graphs

All graphs in this thesis will be undirected, without loops and multiple edges. We say that two vertices  $x$  and  $y$  are *adjacent* if the pair  $\{x, y\}$  is an edge. Such vertices are also called *neighbors* of each other. We say that the graph is *complete* if any two vertices are adjacent, and *empty* if no two vertices are adjacent. The *complement*  $\overline{G}$  of a graph  $G$  is the graph on the same vertices, but with complementary edge set, that is, two vertices are adjacent in  $\overline{G}$  if they are not adjacent in  $G$ . The *degree* (or *valency*) of a vertex is its number of neighbors. If all vertices have the same degree then the graph is called *regular*.

Two graphs are called *isomorphic* if there is a bijection between the respective vertex sets preserving edges. If two graphs are isomorphic, then we shall not distinguish between them. An *automorphism* of a graph is a bijection from the vertex set to itself preserving edges. The set of automorphisms of a graph, with the composition operator, forms a group, called the *automorphism group*.

If  $X$  is a subset of  $V$ , then the induced subgraph of  $G$  on  $X$  is the graph with vertex set  $X$ , and with edges those of  $G$  that are contained in  $X$ . A *coclique* is an induced empty subgraph, and a *clique* is an induced complete subgraph. A graph is called *bipartite* if the vertices can be partitioned into two induced cocliques.

A *walk* of length  $\ell$  between two vertices  $x, y$  is a sequence of (not necessarily distinct) vertices  $x = x_0, x_1, \dots, x_\ell = y$ , such that for any  $i$  ( $0 \leq i \leq \ell - 1$ ) the vertices  $x_i$  and  $x_{i+1}$  are adjacent. If all vertices are distinct then the walk is also called a *path*. If there is a path between any two vertices of the graph, then the graph is called *connected*. The *distance* between two vertices is the length of a shortest path between these vertices.

## 2.2 Spectral characterizations

Throughout this thesis,  $\mathbf{1}$  and  $\mathbf{0}$  shall denote the all-one and the zero vector, respectively. We denote the all-one matrix by  $J$ , the identity matrix by  $I$  and the all-zero matrix by  $O$ .

### 2.2.1 Cospectral graphs

Consider two graphs  $G$  and  $G'$  with adjacency matrices  $A$  and  $A'$ , respectively. As we mentioned before, the graphs  $G$  and  $G'$  are called *cospectral* if  $A$  and  $A'$  have the same spectrum.

For a graph  $G$  with adjacency matrix  $A$ , any matrix of the form  $M = xI + yJ + zA$  with  $x, y, z \in \mathbb{R}$ ,  $z \neq 0$  is called a *generalized adjacency matrix* of  $G$ . Since we are interested in the relation between  $G$  and the spectrum of  $M$ , we can restrict to generalized adjacency matrices of the form  $yJ - A$  without loss of generality. As we shall see in Theorem 1, Johnson and Newman [45] proved that if  $yJ - A$  and  $yJ - A'$  are cospectral for two distinct values of  $y$ , then they are cospectral for all  $y$ , and hence they are cospectral with respect to all generalized adjacency matrices. In this case we will call  $G$  and  $G'$   $\mathbb{R}$ -cospectral. So if  $yJ - A$  and  $yJ - A'$  are cospectral for some but not all values of  $y$ , they are cospectral for exactly one value  $\hat{y}$  of  $y$ . Then we say that  $G$  and  $G'$  are  $\hat{y}$ -cospectral. Thus cospectral graphs (in the usual sense) are either 0-cospectral or  $\mathbb{R}$ -cospectral.

For a graph  $G$  with adjacency matrix  $A$ , the polynomial  $p(x, y) = \det(xI + yJ - A)$  will be called the *generalized characteristic polynomial* of  $A - yJ$ , and  $p(x, 0) = p(x)$  is the characteristic polynomial of  $A$ .

An orthogonal matrix  $Q$  is *regular* if it has constant row sum, that is,  $Q\mathbf{1} = r\mathbf{1}$ .

**Theorem 1.** [45] *If  $G$  and  $G'$  are graphs with adjacency matrices  $A$  and  $A'$ , respectively, then the following are equivalent.*

- i. The graphs  $G$  and  $G'$  are cospectral, and so are their complements.
- ii. The graphs  $G$  and  $G'$  are  $\mathbb{R}$ -cospectral.
- iii. There exists a regular orthogonal matrix  $Q$ , such that  $Q^\top A Q = A'$ .

*Proof.* First, we shall prove that if  $yJ - A$  and  $yJ - A'$  are cospectral for two distinct values of  $y$ , then they are cospectral for all  $y$ , and hence they are cospectral with respect to all generalized adjacency matrices. Let  $G$  and  $G'$  be graphs with generalized characteristic polynomials  $p(x, y)$  and  $p'(x, y)$ , respectively. Note that for fixed  $y$ ,  $p(x, y)$  is the characteristic polynomial of  $A - yJ$ . Since  $J$  has rank 1, the degree in  $y$  of  $p(x, y)$  is 1 (this follows from Gaussian elimination in  $xI + yJ - A$ ), so there exist integers  $a_0, \dots, a_n$  and  $b_0, \dots, b_n$  such that

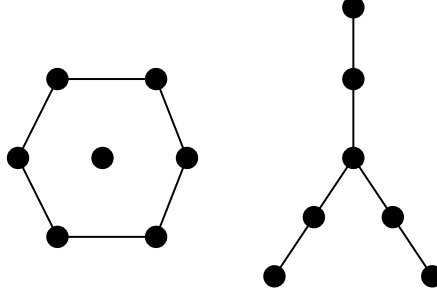
$$p(x, y) = \sum_{i=0}^n (a_i + b_i y) x^i.$$

It is clear that  $p(x, y) \equiv p'(x, y)$  if and only if  $G$  and  $G'$  are  $\mathbb{R}$ -cospectral, and  $G$  and  $G'$  are  $\hat{y}$ -cospectral if and only if  $p(x, \hat{y}) = p'(x, \hat{y})$  for all  $x \in \mathbb{R}$ , whilst  $p(x, y) \not\equiv p'(x, y)$  (indeed, if  $G$  and  $G'$  are  $y$  cospectral for some  $\hat{y}$  but not for all  $y$ , then the corresponding polynomials  $p(x, y)$  and  $p'(x, y)$  are not identical, whilst  $p(x, \hat{y}) = p'(x, \hat{y})$ ). If this is the case, then  $a_i + \hat{y}b_i = a'_i + \hat{y}b'_i$  with  $(a_i, b_i) \neq (a'_i, b'_i)$  for some  $i$  ( $0 \leq i \leq n-3$ ). This implies  $\hat{y} = -(a_i - a'_i)/(b_i - b'_i)$  is unique and rational. Thus we proved the equivalence between i. and ii. Finally, it easily follows that  $G$  and  $G'$  are  $\mathbb{R}$ -cospectral if  $Q$  is regular, since  $Q^\top \mathbf{1} = \mathbf{1}$  implies  $Q^\top(yJ - A)Q = yJ - A'J$ , so  $yJ - A$  and  $yJ - A'$  are cospectral for every  $y \in \mathbb{R}$ . By taking  $y = 1$  we see that  $\mathbb{R}$ -cospectral graphs have cospectral complements.  $\square$

The spectrum of a graph  $G$  together with that of its complement will be referred to as the *generalized spectrum* of  $G$ . We say that a given graph  $G$  is *determined by its spectrum* (DS for short) if every graph cospectral with  $G$  is isomorphic with  $G$ . A graph  $G$  is said to be *determined by its generalized spectrum* (DGS for short) if every graph  $\mathbb{R}$ -cospectral with  $G$  is isomorphic with  $G$ , or equivalently, if every graph cospectral with  $G$  and with complement cospectral to  $\overline{G}$  is isomorphic to  $G$ .

## 2.2.2 Constructing cospectral graphs: GM switching

Many constructions of cospectral graphs are known. Here we focus on one method introduced by Godsil and McKay [33], which seems to be the most



**Figure 2.1** A pair of  $\mathbb{R}$ -cospectral graphs.

productive one. At several points in the rest of the thesis we will make use of it, and it will be referred to simply as GM switching. Godsil and McKay gave the conditions under which the adjacency spectrum is unchanged by this operation.

**Lemma 2.** [33][GM switching] *Let  $G$  be a graph and let  $\{X_1, \dots, X_\ell, Y\}$  be a partition of the vertex set  $V(G)$  of  $G$ . Suppose that for every vertex  $x \in Y$  and every  $i \in \{1, \dots, \ell\}$ ,  $x$  has either  $0$ ,  $\frac{1}{2}|X_i|$  or  $|X_i|$  neighbors in  $X_i$ . Moreover, suppose that for all  $i, j \in \{1, \dots, \ell\}$  the number of neighbors of an arbitrary vertex of  $X_i$  that are contained in  $X_j$ , depends only on  $i$  and  $j$  and not on the vertex. Make a new graph  $G'$  from  $G$  as follows. For each  $x \in Y$  and  $i \in \{1, \dots, \ell\}$  such that  $x$  has  $\frac{1}{2}|X_i|$  neighbors in  $X_i$  delete the corresponding  $\frac{1}{2}|X_i|$  edges and join  $x$  instead to the  $\frac{1}{2}|X_i|$  other vertices in  $X_i$ . Then  $G$  and  $G'$  are cospectral (with cospectral complements).*

See Section 3.2 for a proof.

The operation that changes  $G$  into  $G'$  is called *Godsil-McKay switching*. Note that the pair of graphs in Figure 2.1 is related by GM switching ( $\ell = 1$  and  $X_1$  is a 4-coclique), and hence has cospectral complements. The pair of graphs in Figure 1.1 does not have cospectral complements and hence does not arise by GM switching.

If  $\ell = 1$  and  $|X_1| = 2$ , then GM switching interchange the two vertices in  $X_1$ , so  $G$  and  $G'$  are isomorphic, and we call the switching trivial. But if  $\ell = 1$  and  $|X_1| \geq 4$ , then GM switching usually produces nonisomorphic graphs [42].

### 2.2.3 Computer results

The paper [33] by Godsil and McKay also gives interesting computational results for cospectral graphs. In particular, they generate and check cospectrality in all graphs up to 9 vertices. This enumeration has been extended to 11 vertices by Haemers and Spence [42], and cospectrality was tested with respect to the adjacency matrix  $A$ , the set of generalized adjacency matrices  $A \& \bar{A}$ , the Laplacian matrix  $L = D - A$ , and the signless Laplacian matrix  $Q = D + A$  ( $D$  is the diagonal matrix with the degrees). The results are in Table 2.1 [14], where the fractions of non-DS graphs for each of the four cases are given. GM switching also works for  $L$  and  $Q$ , but then the conditions are not the same as in Lemma 2, see [60] for details. The last three columns give the fractions of graphs for which GM switching gives cospectral nonisomorphic graphs with respect to  $A$ ,  $L$  and  $Q$ , respectively.

$n$	# graphs	$A$	$A \& \bar{A}$	$L$	$Q$	GM- $A$	GM- $L$	GM- $Q$
2	2	0	0	0	0	0	0	0
3	4	0	0	0	0	0	0	0
4	11	0	0	0	0.182	0	0	0
5	34	0.059	0	0	0.118	0	0	0
6	156	0.064	0	0.026	0.103	0	0	0
7	1044	0.105	0.038	0.125	0.098	0.038	0.069	0
8	12346	0.139	0.094	0.143	0.097	0.085	0.088	0
9	274668	0.186	0.160	0.155	0.069	0.139	0.110	0
10	12005168	0.213	0.201	0.118	0.053	0.171	0.080	0.001
11	1018997864	0.211	0.208	0.090	0.038	0.174	0.060	0.001
12	165091172592	0.188		0.060	0.027			

**Table 2.1** Fractions of non-DS graphs.

Note that Table 2.1 indicates that at least for some small graphs the signless Laplacian may be a better matrix for spectral characterizations than the adjacency or the Laplacian matrix.

Notice also that for  $n \leq 4$  there are no cospectral graphs with respect to  $A$  or to  $L$ , but there is one such pair with respect to  $Q$ , namely  $K_{1,3}$  and  $K_1 + K_3$ . For  $n = 5$  there is just one pair with respect to  $A$ :  $K_{1,4}$  and  $K_1 + C_4$  (see Figure 1.1).

An interesting result from the table is that the fraction of non-DS graphs is nondecreasing for small  $n$ , but starts to decrease at  $n = 10$  for  $A$ , at  $n = 9$  for  $L$ , and at  $n = 6$  for  $Q$ . Especially for the Laplacian matrix and signless Laplacian matrix, these data suggest that the fraction of non-DS graphs might tend to 0 as  $n \rightarrow \infty$ . In addition, the table shows that the majority of non-DS graphs with respect to  $A$  and  $L$  comes from GM switching (at least for  $n \geq 7$ ). If this tendency continues, almost all graphs would be DS for all three cases. Indeed, the fraction of graphs that admit a nontrivial GM switching tends to zero as  $n$  tends to infinity, and the partitions with  $\ell = 1$  and  $|X_1| = 4$  account for most of these switchings (see also [33]). For data for  $n = 12$ , see [15].

#### 2.2.4 The method of Wang and Xu

In [69, 70], Wang and Xu gave a method for determining whether a graph  $G$  is determined by its generalized spectrum (DGS), which works for a large family of general graphs. Their key observation is the following:

Let  $G$  and  $G'$  be two graphs that are cospectral with cospectral complements. Let  $A$  and  $A'$  be the adjacency matrices of  $G$  and  $G'$ , respectively. Thus, if it can be shown that every regular orthogonal matrix  $Q$  such that  $Q^T A Q$  is a  $(0, 1)$  matrix must be a permutation matrix, then  $G$  is clearly DGS.

In [67], Wang continued this line of research reviewing some of the previous results and improving the results in [69, 70]. Let  $G$  be a graph on  $n$  vertices with adjacency matrix  $A$ . The *walk matrix*  $W$  of  $G$  is the square matrix of order  $n$  with  $i$ -th column  $A^{i-1} \mathbf{1}$  ( $1 \leq i \leq n$ ). In [67], Wang shows that the DGS property of a graph can often be deduced from the prime factorization of the determinant of the walk-matrix  $\det(W)$ . In particular, the author defines a large family of graphs  $\mathcal{F}_n$  that consists of graphs  $G$  with  $\frac{\det(W)}{2^{\lfloor \frac{n}{2} \rfloor}}$  being odd and square-free, and shows that every graph in  $\mathcal{F}_n$  is DGS.

If every regular orthogonal matrix  $Q$  for which  $Q^T A Q$  is a  $(0, 1)$  matrix is a permutation matrix, then  $G$  is DGS. At first glance, this approach seems as difficult as the original problem. However, recently Wang [68] managed to find some algorithmic methods to achieve this goal by using some arithmetic properties of the walk-matrix associated with the given graph.

Below we give some ideas of the method of Wang and Xu [70]. For more details about this method, see the textbook of Brouwer and Haemers [14, §14.6].

The walk matrix  $W$  of a graph with adjacency matrix  $A$  is nonsingular if and only if  $A$  does not have an eigenvector orthogonal to  $\mathbf{1}$ . Note that the walk matrix of a regular graph is singular. Assume that  $G$  and  $G'$  are cospectral with cospectral complements. Call their walk matrices  $W$  and  $W'$ . Wang and Xu proved that if  $W$  is nonsingular, then  $W'$  is nonsingular, and  $Q = W'W^{-1}$  is the unique regular orthogonal matrix such that  $A' = Q^\top A Q$ . In particular,  $Q$  is rational.

This reduces the problem to studying rational matrices  $Q$  with  $QQ^\top = I$ ,  $Q\mathbf{1} = \mathbf{1}$  and  $Q^\top A Q$  a  $(0,1)$  matrix. Recall that  $Q$  is said to have *level*  $l$  whenever  $l$  is the smallest positive integer such that  $lQ$  is an integral matrix. If  $Q$  has level 1, then it is a permutation matrix, and  $G$  and  $G'$  are isomorphic graphs. So the graph  $G$  with nonsingular walk matrix  $W$  is determined by its spectrum and the spectrum of its complement when all such matrices  $Q$  have level 1. If  $Q$  has level 2, we call  $G$  and  $G'$  semi-isomorphic graphs. Also, the regular orthogonal matrices of level 2 are classified; this is the starting point of Section 3.3.

There is experimental evidence that in most cases where a nonisomorphic cospectral mate exists, the level  $l$  is 2 (leading to semi-isomorphic graphs).

## 2.3 Interlacing

In this section we introduce an important spectral technique: eigenvalue interlacing.

Consider two sequences of real numbers:  $\lambda_1 \geq \dots \geq \lambda_n$  and  $\mu_1 \geq \dots \geq \mu_m$  with  $m < n$ . The second sequence is said to *interlace* the first one whenever

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{for } i = 1, \dots, m.$$

The interlacing is called *tight* if there exist an integer  $k \in [0, m]$  such that

$$\lambda_i = \mu_i \text{ for } 1 \leq i \leq k \quad \text{and} \quad \lambda_{n-m+i} = \mu_i \text{ for } k+1 \leq i \leq m.$$

If  $m = n-1$ , the interlacing inequalities become  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_m \geq \lambda_n$ , which clarifies the name. Throughout, the  $\lambda_i$ s and the  $\mu_i$ s will be



eigenvalues of matrices  $A$  and  $B$ , respectively.

**Theorem 3.** [37][Interlacing] *Let  $A$  be a real symmetric  $n \times n$  matrix with eigenvalues  $\lambda_1 \geq \dots \geq \lambda_n$ . For some  $m < n$ , let  $S$  be a real  $n \times m$  matrix with orthonormal columns,  $S^\top S = I$ , and consider the matrix  $B = S^\top A S$ , with eigenvalues  $\mu_1 \geq \dots \geq \mu_m$ . Then,*

(a) *the eigenvalues of  $B$  interlace those of  $A$ , that is,*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}, \quad i = 1, \dots, m, \quad (2.1)$$

(b) *if the interlacing is tight, then  $SB = AS$ .*

Two interesting particular cases of interlacing are obtained by choosing appropriately the matrix  $S$ .

If  $S = [I \ O]^\top$ , then  $B$  is just a principal submatrix of  $A$  and we have:

**Corollary 4.** *If  $B$  is a principal submatrix of a symmetric matrix  $A$ , then the eigenvalues of  $B$  interlace the eigenvalues of  $A$ .*

If  $\mathcal{P} = \{U_1, \dots, U_m\}$  is a partition of the vertex set  $V$ , with each  $U_i \neq \emptyset$ , we can take for  $\tilde{B}$  the so-called quotient matrix of  $A$  with respect to  $\mathcal{P}$ . Let  $A$  be partitioned according to  $\mathcal{P}$ :

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{bmatrix},$$

where  $A_{i,j}$  denotes the submatrix (block) of  $A$  formed by rows in  $U_i$  and columns in  $U_j$ . The *characteristic matrix*  $C$  is the  $n \times m$  matrix whose  $j^{\text{th}}$  column is the characteristic vector of  $U_j$  ( $j = 1, \dots, m$ ).

Then, the *quotient matrix* of  $A$  with respect to  $\mathcal{P}$  is the  $m \times m$  matrix  $\tilde{B}$  whose entries are the average row sums of the blocks of  $A$ , more precisely:

$$(\tilde{B})_{i,j} = \frac{1}{|U_i|} \mathbf{1}^\top A_{i,j} \mathbf{1} = \frac{1}{|U_i|} (C^\top A C)_{i,j}.$$

The partition is called *equitable* (or *regular*) if each block  $A_{i,j}$  of  $A$  has constant row (and column) sum, that is,  $C\tilde{B} = AC$ .

**Corollary 5.** *Suppose  $\tilde{B}$  is the quotient matrix of a symmetric partitioned matrix  $A$ .*

- (i) *The eigenvalues of  $\tilde{B}$  interlace the eigenvalues of  $A$ .*
- (ii) *If the interlacing is tight then the partition is regular.*

*Proof.* Take  $D = \text{diag}(|U_1|, \dots, |U_m|) = C^\top C$ ,  $S = CD^{-1/2}$  and  $B = S^\top AS$ . Then, since  $B = D^{1/2} \tilde{B} D^{-1/2}$ ,  $B$  and  $\tilde{B} = D^{-1/2} B D^{1/2}$  have the same spectrum, and the eigenvalues of  $B = S^\top AS$  interlace those of  $A$ , which proves (i). If the interlacing is tight, then  $SB = AS$ ; hence,  $C\tilde{B} = AC$ .  $\square$

Note that  $\tilde{B}$  need not to be a symmetric matrix. However, the proof of Corollary 5 shows that  $\tilde{B}$  is diagonally similar to  $B$ , which is symmetric.

Note also that the converse of Corollary 5.(ii) is not true: a regular partition does not imply tight interlacing. Take, for example, the cube graph  $Q_3$ , with spectrum of the adjacency matrix  $\{3, 1^3, -1^3, -3\}$ . If we consider the partition of the hypercube into antipodal pairs of vertices we get a  $4 \times 4$  quotient matrix  $\tilde{B}$  with spectrum  $\{3, -1^3\}$ . Thus, the smallest eigenvalues of  $\tilde{B}$  and  $A$  are not equal, so there is not tight interlacing.

## 2.4 Distance-regular graphs

A connected graph  $G$  with diameter  $d$  is called *distance-regular* with intersection array

$$\iota(G) = \{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\},$$

if it is regular of degree  $k$ , and if for any two vertices  $u, v$  at distance  $i$ , there are precisely  $c_i$  neighbors of  $v$  at distance  $i-1$  from  $u$ , and  $b_i$  neighbors of  $v$  at distance  $i+1$  from  $u$ . The numbers  $c_i$ ,  $b_i$  and  $a_i$ , where  $a_i = k - b_i - c_i$  is the number of neighbors of  $v$  at distance  $i$  from  $u$ , are called the *intersection numbers* of  $G$ . By definition we have that

$$b_0 = k, \quad b_d = c_0 = 0, \quad c_1 = 1.$$

An intuitive way of looking at distance-regularity is to “hang” the graph from a given vertex and observe the resulting different “layers” in which the vertex set is partitioned; that is, the subsets of vertices at given distances from

the root: if vertices in the same layer are “neighbourhood-indistinguishable” from each other, and the whole configuration does not depend on the chosen vertex, the graph is distance-regular. More formally, a graph is called distance-regular if for every vertex  $u$  there is an equitable partition of the vertices, with quotient matrix being the same for every  $u$ .

A generalization of the adjacency matrix, which is very useful in the study of distance-regular graphs, is the concept of distance matrix  $A_i$ . For every  $i = 0, \dots, d$ , the distance matrix  $A_i$  has entries  $(A_i)_{uv} = 1$  if the distance between  $u$  and  $v$ , denoted  $\text{dist}(u, v)$ , is given by  $\text{dist}(u, v) = i$ , and  $(A_i)_{uv} = 0$  otherwise. One easily checks that the matrices  $A_i$  of a distance-regular graph satisfy the relations

$$\begin{aligned} A_0 &= I, \quad A_1 = A, \\ AA_i &= c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1} \quad (i = 1, 2, \dots, d), \\ A_0 + A_1 + \dots + A_d &= J. \end{aligned}$$

The *intersection matrix*  $B$  (or the quotient matrix of the distance partition) of  $G$  is the tridiagonal matrix

$$B = \begin{bmatrix} a_0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & b_2 & & \\ & & c_3 & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & b_{d-1} \\ & & & & & c_d & a_d \end{bmatrix}.$$

Clearly the intersection array determines the intersection matrix. Some trivial examples of distance-regular graphs are the complete graphs (case  $d = 1$ ), and the polygons (case  $k = 2$ ). In both of these cases the graphs are characterized by their spectra. A distance-regular graph with diameter 2 is the same as a connected strongly regular graph.

# Cospectral graphs and regular orthogonal matrices of level 2

---

In this chapter we show how semi-isomorphic graphs can be constructed by a switching procedure, that generalizes the switching method due to Godsil and McKay [33]. We start with the classification of indecomposable regular orthogonal matrices of level 2, and then consider the generalized switching for the case that  $Q$  has one nontrivial indecomposable block of order 4, 6, 7 or 8. In terms of the graph  $G$  it means that  $G$  must have a subgraph  $\Delta$  of one of the mentioned orders that satisfies a number of properties. The four vertex case corresponds to GM switching and the required properties are easily described; see Section 2.2.2. If  $\Delta$  has six or seven vertices the required properties are worked out in detail. For eight vertices we restrict to the case  $\Delta = G$ .

As an application we determine all new switchings for graphs with eight vertices. We find 68 graphs for which GM switching does not work, but the new switching does. It turns out that there exist only 22 pairs of  $\mathbb{R}$ -cospectral graphs on eight vertices which are not semi-isomorphic with each other or with another graph.

## 3.1 Preliminaries

Recall that an orthogonal matrix  $Q$  is *regular* if it has constant row sum, that is,  $Q\mathbf{1} = r\mathbf{1}$ . From  $Q^\top Q = QQ^\top = I$ , it follows that also  $Q^\top \mathbf{1} = r\mathbf{1}$ , and

that  $r = \pm 1$ . Without loss of generality we will assume  $r = 1$ . A regular orthogonal matrix  $Q$  has *level*  $l$  if  $l$  is the smallest positive integer such that  $lQ$  is an integral matrix. We define  $l = \infty$  if  $Q$  has irrational entries. Clearly  $l = 1$  if and only if  $Q$  is a permutation matrix.

As we saw in Section 2, since  $A$  and  $A'$  are symmetric,  $G$  and  $G'$  are cospectral if and only if  $A$  and  $A'$  are similar, that is, there exists an orthogonal matrix  $Q$  such that  $Q^\top A Q = A'$ . If  $Q$  is a permutation matrix (i.e.  $Q$  is regular of level 1) then  $G$  and  $G'$  are isomorphic. If  $G$  and  $G'$  are nonisomorphic, and there exist a regular orthogonal matrix  $Q$  of level 2 such that  $Q^\top A Q = A'$ , we call  $G$  and  $G'$  *semi-isomorphic*. It easily follows that  $G$  and  $G'$  are  $\mathbb{R}$ -cospectral if  $Q$  is regular. (Indeed,  $Q^\top \mathbf{1} = \mathbf{1}$  implies  $Q^\top (yJ - A)Q = yJ - A'$ ). In particular, semi-isomorphic graphs are  $\mathbb{R}$ -cospectral. By taking  $y = 1$  we see that  $\mathbb{R}$ -cospectral graphs have cospectral complements. Note that Theorem 1 states that the converse of some of these observations is also true.

### 3.2 Switching

We start with a proof of the GM switching, since it shows the use of regular orthogonal matrices. For convenience, we repeat Lemma 2.

**Lemma 6.** *Let  $G$  be a graph and let  $\{X_1, \dots, X_\ell, Y\}$  be a partition of the vertex set  $V(G)$  of  $G$ . Suppose that for every vertex  $x \in Y$  and every  $i \in \{1, \dots, \ell\}$ ,  $x$  has either 0,  $\frac{1}{2}|X_i|$  or  $|X_i|$  neighbors in  $X_i$ . Moreover, suppose that for all  $i, j \in \{1, \dots, \ell\}$  every vertex  $x \in X_i$  has the same number of neighbors in  $X_j$ . Make a new graph  $G'$  as follows. For each  $x \in Y$  and  $i \in \{1, \dots, \ell\}$  such that  $x$  has  $\frac{1}{2}|X_i|$  neighbors in  $X_i$  delete the corresponding  $\frac{1}{2}|X_i|$  edges and join  $x$  instead to the  $\frac{1}{2}|X_i|$  other vertices in  $X_i$ . Then  $G$  and  $G'$  are  $\mathbb{R}$ -cospectral.*

*Proof.* Let  $A$  and  $A'$  be the adjacency matrices of  $G$  and  $G'$ , respectively (the vertex ordering is assumed to be in accordance with the partition). Let  $n$  be the number of vertices of  $G$  and  $G'$ . For  $i = 1, \dots, \ell$  define the  $|X_i| \times |X_i|$  matrix  $R_i = \frac{2}{|X_i|}J - I$ , and the  $n \times n$  block diagonal matrix  $Q = \text{diag}(R_1, \dots, R_\ell, I)$ . Then  $Q$  is orthogonal and regular, and it follows straightforwardly that  $Q^\top A Q = A'$ , and more generally, that  $Q^\top (yJ - A)Q = yJ - A'$  for every  $y \in \mathbb{R}$ .  $\square$

Note that the orthogonal matrix  $Q$  used in the proof of the Godsil-McKay switching is regular of level  $\text{lcm}(|X_1|, \dots, |X_\ell|)/2$ . If  $|X_i| = 2$  for some  $i \in \{1, \dots, \ell\}$ , then GM switching just interchanges the two vertices of  $X_i$ , and therefore the two vertices may be considered part of  $Y$ . Thus we can assume that  $|X_i| \geq 4$ . If  $|X_i| = 4$  for all  $i \in \{1, \dots, \ell\}$ , then  $Q$  has level 2, and the graphs  $G$  and  $G'$  are semi-isomorphic, provided they are not isomorphic. The conditions for GM switching are easiest to fulfill if  $\ell = 1$  and  $|X_1| = 4$ . In this case the orthogonal matrix  $Q$  is regular of level 2 and has just one nontrivial indecomposable block  $R_1 = \frac{1}{2}J - I$ . For this switching to work,  $X_1$  must induce a regular graph on four vertices, and each vertex outside  $X_1$  should be adjacent to 0, 2, or 4 vertices of  $X_1$ . For example, the adjacency matrix  $A$  given below satisfies these conditions, and  $A'$  is obtained by GM switching:  $A' = Q^\top A Q$ . Therefore the two graphs are  $\mathbb{R}$ -cospectral. The graphs are not isomorphic (because of different degree sequences), and therefore they are semi-isomorphic.

$$A = \left[ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \end{array} \right], \quad A' = \left[ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \end{array} \right].$$

This is the situation we will generalize. If  $R$  is an indecomposable regular orthogonal  $r \times r$  matrix of level 2, and  $G$  is a graph with  $n \geq r$  vertices and adjacency matrix  $A$ . We define the  $n \times n$  matrix

$$Q = \begin{bmatrix} R & O \\ O & I \end{bmatrix}$$

and investigate the required structure for  $A$  needed to ensure that  $A' = Q^\top A Q$  is again the adjacency matrix of a graph. Note that it is sufficient to require that  $A'$  is a  $(0, 1)$  matrix, because  $A'$  is symmetric and  $\text{trace } A' = \text{trace } A = 0$ .

### 3.3 Regular orthogonal matrices of level 2

Let  $Q$  be a regular orthogonal matrix of level 2. Then after suitable reordering of rows and columns,  $Q$  takes the block diagonal form  $\text{diag}(R_1, \dots, R_\ell)$ , or  $\text{diag}(R_1, \dots, R_\ell, I)$ , where  $R_i$  is an indecomposable regular orthogonal

matrix of level 2 for  $i = 1, \dots, \ell$ . It follows easily that if  $R$  is an indecomposable regular orthogonal matrix of level 2, then all entries of  $2R$  are equal to 0, 1 or  $-1$ , and each row and column of  $R$  has exactly three 1's and one  $-1$ . Using these observations and the orthogonality of  $R$ , Wang and Xu [69, 71] determined all indecomposable regular orthogonal matrices of level 2.

**Theorem 7.** *Let  $R$  be an indecomposable regular orthogonal matrix with level 2 and row sum 1. Then after suitable reordering of rows and columns  $R$  is one of the following:*

$$(i) \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \quad (ii) \frac{1}{2} \begin{bmatrix} J & O & \dots & \dots & O & Y \\ Y & J & O & \dots & \dots & O \\ O & Y & J & O & \dots & O \\ & \ddots & \ddots & \ddots & \ddots & \\ O & \dots & O & Y & J & O \\ O & \dots & \dots & O & Y & J \end{bmatrix},$$

$$(iii) \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}, \quad (iv) \frac{1}{2} \begin{bmatrix} -I & I & I & I \\ I & -Z & I & Z \\ I & Z & -Z & I \\ I & I & Z & -Z \end{bmatrix},$$

where  $I, J, O, Y = 2I - J$  and  $Z = J - I$ , are square matrices of order 2.

We observed that  $W = 2R$  is a matrix with entries 0, 1 and  $-1$ , satisfying  $WW^\top = 4I$ , and  $W\mathbf{1} = W^\top\mathbf{1} = 2\cdot\mathbf{1}$ . Such a matrix  $W$  is known as a regular weighing matrix of weight 4. Two weighing matrices are called equivalent if one can be obtained by the other by row and column permutations and/or multiplication of a number of rows and columns by  $-1$ . The inequivalent weighing matrices of weight 4 have been classified in 1986 by Chan, Rodger and Seberry [18], and the classification of the regular ones follows from their result. Therefore, Theorem 7 should be attributed to the authors of [18].

Case (ii) of the above theorem gives an infinite family of matrices of even order starting with order 6. So for the order 8 there exist two different indecomposable regular orthogonal matrices of level 2. If

$$Q = \begin{bmatrix} R & O \\ O & I \end{bmatrix}$$

and  $R$  is as in case (i), then the transformation  $A' = Q^\top A Q$  corresponds to GM switching. In the next sections we will investigate the required structure for  $A$  for the other three cases.

The product of two regular orthogonal matrices of level 2 is again a regular orthogonal matrix, but the level need not be 2, but can also be 1 or 4. Therefore we may not conclude that the relation: ‘being isomorphic or semi-isomorphic’ is an equivalence relation. In fact, this is false. This is illustrated by the following example.

**Example 8.** Consider the three nonisomorphic graphs  $G, G_1, G_2$  with adjacency matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The graphs  $G_1$  and  $G_2$  can both be obtained from  $G$  by GM switching. Therefore  $G_1$  and  $G_2$  are both semi-isomorphic with  $G$ . The regular orthogonal matrices that represent the switching are (with  $R$  as in Case (i) of Theorem 7):

$$Q_1 = Q_1^\top = \begin{bmatrix} R & O \\ O & I_4 \end{bmatrix}, \text{ and } Q_2 = Q_2^\top = \begin{bmatrix} I_3 & O & \mathbf{0} \\ O & R & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & 1 \end{bmatrix}.$$

Clearly,  $Q = Q_1 Q_2$  is orthogonal and regular and satisfies  $Q^\top A_1 Q = A_2$ . But  $Q$  has level 4. Moreover, it has been checked (by computer) that there exists no orthogonal regular  $Q$  of level 2 for which  $Q^\top A_1 Q = A_2$ . Therefore,  $G_1$  and  $G_2$  are not semi-isomorphic.

In some cases the product of two regular orthogonal matrices  $Q_1$  and  $Q_2$  of level 2 has level 2 again. This is obviously the case, if the rows of the nontrivial indecomposable blocks of  $Q_1$  are all different from the rows of the nontrivial indecomposable blocks of  $Q_2$ . A nontrivial example is given by:

$$Q_1 = Q_1^\top = \begin{bmatrix} R_2 & O \\ O & I_2 \end{bmatrix}, \text{ and } Q_2 = Q_2^\top = \begin{bmatrix} I & O \\ O & R_1 \end{bmatrix},$$



with  $R_1$  as in Case (i), and  $R_2$  as in Case (i) or (ii) of Theorem 7. Then  $Q_1Q_2$  has again level 2 and belongs to Case (ii) of Theorem 7. In case both  $R_1$  and  $R_2$  belong to Case (i), then  $Q_1Q_2$  correspond to a six vertex switching of Case (ii). This shows that the six vertex switching can sometimes be obtained by applying GM switching twice.

### 3.4 Six vertex switching

Here we consider switching with a regular orthogonal matrix  $Q$  of order  $n$ , having just one nontrivial indecomposable block of order 6. Thus with a suitable ordering of rows and columns we have:

$$Q = \begin{bmatrix} R & O \\ O & I \end{bmatrix}, \text{ where } R = \frac{1}{2} \begin{bmatrix} J & O & Y \\ Y & J & O \\ O & Y & J \end{bmatrix}, \text{ and } Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Let  $G$  be a graph with  $n$  vertices and adjacency matrix

$$A = \begin{bmatrix} B & V \\ V^\top & C \end{bmatrix},$$

where  $B$  is the adjacency matrix of a graph  $\Delta$  of order 6. For the six vertex switching with respect to  $\Delta$  to work we need that the switched matrix

$$A' = Q^\top A Q = \begin{bmatrix} R^\top B R & R^\top V \\ V^\top R & C \end{bmatrix}$$

is a  $(0,1)$  matrix again. First we determine the possible columns of  $V$ . This means that we have to find the vectors  $\mathbf{v} \in \{0,1\}^6$  for which  $R^\top \mathbf{v}$  is again a  $(0,1)$  vector.

**Lemma 9.** *Let  $\mathbf{v} \in \{0,1\}^6$ . With  $R$  as above,  $R^\top \mathbf{v} \in \{0,1\}^6$  if and only if the number of ones in each class of the partition is even, or the number of ones in each class of the partition is odd. In the first case  $R^\top \mathbf{v} = \mathbf{v}$ . In the second case, multiplication by  $R^\top$  gives a permutation of the eight involved  $(0,1)$  vectors represented by the following two cycles  $([101010]^\top \text{ and } [010101]^\top \text{ are fixed})$ :*

$$\left( [101001]^\top, [100110]^\top, [011010]^\top \right) \left( [100101]^\top, [010110]^\top, [011001]^\top \right).$$

*Proof.* With  $\mathbf{v} = [v_1 \dots v_6]^\top$  we have

$$\mathbf{0} = 2R^\top \mathbf{v} = \begin{bmatrix} v_1 + v_2 + v_3 - v_4 \\ v_1 + v_2 - v_3 + v_4 \\ v_3 + v_4 + v_5 - v_6 \\ v_3 + v_4 - v_5 + v_6 \\ v_1 - v_2 + v_5 + v_6 \\ -v_1 + v_2 + v_5 + v_6 \end{bmatrix} = \begin{bmatrix} v_{1,2} + v_{3,4} \\ v_{1,2} + v_{3,4} \\ v_{3,4} + v_{5,6} \\ v_{3,4} + v_{5,6} \\ v_{1,2} + v_{5,6} \\ v_{1,2} + v_{5,6} \end{bmatrix} \pmod{2},$$

where  $v_{i,i+1} = v_i + v_{i+1}$  for  $i = 1, 3, 5$ . It follows that  $R^\top \mathbf{v}$  is a  $(0, 1)$  vector if and only if  $v_{1,2} = v_{3,4} = v_{5,6} \pmod{2}$ . The second part of the lemma follows by straightforward verification.  $\square$

Next we determine the set  $\mathcal{B}$  of adjacency matrices  $B$  of order 6, that have the property that  $B' = R^\top B R$  is a  $(0, 1)$  matrix again. To do so, the following observations are useful. The matrix  $R$  is invariant under certain reorderings of rows and columns, more precisely:  $R = P^\top R P$ , when  $P$  is any permutation matrix generated by

$$P_1 = \begin{bmatrix} O & I & O \\ O & O & I \\ I & O & O \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} Z & O & O \\ O & Z & O \\ O & O & Z \end{bmatrix}, \text{ where } Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly,  $B' = R^\top B R$  implies  $P^\top B' P = R^\top (P^\top B P) R$ , so  $\mathcal{B}$  is invariant under the mentioned permutations and  $(P^\top B P)' = P^\top B' P$ . Moreover,  $B' = R^\top B R$  implies  $J - B' - I = R^\top (J - B - I) R$ , so  $\mathcal{B}$  is also invariant under taking complements and  $(J - B - I)' = J - B' - I$ . But there is more. The permutation matrix  $P_2$  commutes with  $R$ , and therefore  $P_2 + B' = R^\top (P_2 + B) R$ , so if  $B \in \mathcal{B}$ , and the three diagonal blocks of  $B$  are  $O$ , then  $B + P_2 \in \mathcal{B}$  and  $(P_2 + B)' = P_2 + B'$ .

**Lemma 10.** *Let  $B$  be an adjacency matrix of order six. With  $R$  as above, the matrix  $B' = R^\top B R$  is again an adjacency matrix if and only if  $B$  can be obtained from one of the following  $B_0 \dots B_7$  by the above mentioned opera-*

tions.

$$\begin{aligned}
 B_0 = O, B_1 &= \begin{bmatrix} O & J & O \\ J & O & O \\ O & O & O \end{bmatrix}, B_2 = \begin{bmatrix} O & I & I \\ I & O & I \\ I & I & O \end{bmatrix}, B_3 = \begin{bmatrix} O & I & I \\ I & O & Z \\ I & Z & O \end{bmatrix}, \\
 B_4 &= \begin{bmatrix} O & N & N^\top \\ N^\top & O & N \\ N & N^\top & O \end{bmatrix}, B_5 = \begin{bmatrix} O & M & N^\top \\ M^\top & O & N \\ N & N^\top & O \end{bmatrix}, B_6 = \begin{bmatrix} O & O & I \\ O & O & M \\ I & M^\top & O \end{bmatrix}, \\
 B_7 &= \begin{bmatrix} O & O & Z \\ O & O & N \\ Z & N^\top & O \end{bmatrix},
 \end{aligned}$$

where  $N = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ ,  $M = J - N$  and  $Z = J - I$ . The switched matrices  $B'_i = R^\top B_i R$  are:

$$\begin{aligned}
 B'_0 = O, B'_1 &= B_1, B'_2 = B_2, B'_3 = \begin{bmatrix} O & I & Z \\ I & O & I \\ Z & I & O \end{bmatrix}, \\
 B'_4 &= \begin{bmatrix} O & N^\top & N \\ N & O & N^\top \\ N^\top & N & O \end{bmatrix}, B'_5 = \begin{bmatrix} O & N^\top & N \\ N & O & M^\top \\ N^\top & M & O \end{bmatrix}, B'_6 = \begin{bmatrix} O & O & M \\ O & O & I \\ M^\top & I & O \end{bmatrix}, \\
 B'_7 &= \begin{bmatrix} O & O & N \\ O & O & Z \\ N^\top & Z & O \end{bmatrix}.
 \end{aligned}$$

*Proof.* With the vertex ordering used for  $R$ , we write

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix}, \text{ and } B' = R^\top B R = \begin{bmatrix} B'_{1,1} & B'_{1,2} & B'_{1,3} \\ B'_{2,1} & B'_{2,2} & B'_{2,3} \\ B'_{3,1} & B'_{3,2} & B'_{3,3} \end{bmatrix}.$$

This leads to

$$4B'_{i,i} = JB_{i,i}J + JB_{i,i+1}Y + YB_{i+1,i}J + YB_{i+1,i+1}Y, \quad (3.1)$$

for  $i = 1, 2, 3$  (addition mod 3), where  $B_{i,j} = B_{j,i}^\top$ . Without loss of generality we take  $B_{1,1} = O$ . Taking traces in Equation 3.1 yields  $\text{trace}(YB_{2,2}Y) = 0$ ,

and therefore  $B_{2,2} = O$ . Thus  $B_{i,i} = O$  for  $i = 1, 2, 3$ . Equation 3.1 becomes  $4B'_{i,i} = JB_{i,i+1}Y + (JB_{i,i+1}Y)^\top$ . For every  $2 \times 2$  matrix  $X$ ,  $JXY = \alpha(M - N)$  for some scalar  $\alpha$ . Since  $B'_{i,i}$  has no negative entries it follows that  $\alpha = 0$  when  $X = B_{i,i+1}$ . Therefore  $JB_{i,i+1}Y = O$ , which reflects that  $B_{i,i+1}$  has constant column sums for  $i = 1, 2, 3$ . Equivalently,  $B_{i,i+2} = B_{i+2,i}^\top = B_{i+2,i+3}^\top$  has constant row sums for  $i = 1, 2, 3$ . Now it is straightforward to find all admissible matrices  $B$  and the corresponding  $B'$ .  $\square$

For example the following matrix  $A$  has the desired form (indeed,  $B = B_4 + P_2$  and  $V$  has columns  $[001100]^\top$  and  $[101001]^\top$ ). With the above lemmas we conclude that the switched matrix  $A'$  is cospectral with  $A$ .

$$A = \left[ \begin{array}{cccccc|cc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array} \right], \quad A' = \left[ \begin{array}{cccccc|cc} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right].$$

The two graphs are not isomorphic, because the degree sequences differ, but they are semi-isomorphic. In addition, it has been verified by computer that the graphs are not related by GM switching.

Out of the eight adjacency matrices presented in Lemma 10, the graphs with matrices  $B_4$  and  $B_5$  are isomorphic, and the same is true for  $B_6$  and  $B_7$ . In addition, the complement of  $B_4$  (and  $B_5$ ) is isomorphic with  $B_4 + P_2$ , and the complement of  $B_2$  is isomorphic with  $B_3 + P_2$ . Therefore, the total number of nonisomorphic graphs  $\Delta$  for which the six vertex switching works is 18. The total number of matrices  $B$  for which  $R^\top B R$  is a  $(0, 1)$  matrix equals 96.

We note that in Lemma 10 in all cases the graph  $\Delta'$  with matrix  $B'$  is isomorphic to  $\Delta$  with matrix  $B$ . This implies that with a suitable reordering of the rows and columns of  $R$  we can establish that  $B' = B$ . However, this would require a reordering of the entries of the vectors in Lemma 9 depending on the choice of  $B$ . So it would not have made the presentation easier. Besides that, the phenomenon is not general, as we shall see in the next section.

### 3.5 Seven vertex switching

Here we consider switching with a regular orthogonal matrix  $Q$  of order  $n$ , having just one nontrivial indecomposable block  $R$  of order 7. Theorem 7 gives

$$R = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

Let  $G$  be a graph with  $n$  vertices and adjacency matrix

$$A = \begin{bmatrix} B & V \\ V^\top & C \end{bmatrix},$$

where now  $B$  is the adjacency matrix of a graph  $\Delta$  with seven vertices. For the seven vertex switching with respect to  $\Delta$  to work we need that the switched matrix

$$A' = Q^\top A Q = \begin{bmatrix} R^\top B R & R^\top V \\ V^\top R & C \end{bmatrix}$$

is a  $(0,1)$  matrix again. Note that the matrix  $R$  is invariant under a cyclic shift, that is,  $P_1 R P_1^\top = R$  for the cyclic permutation matrix  $P_1 = \text{cycle}(0,1,0,0,0,0,0)$ . Moreover, also the following permutation leaves  $R$  unchanged:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus the permutation group  $\mathcal{G}$  generated by  $P_1$  and  $P_2$  is an automorphism group of  $R$ .

**Remark.** The group  $\mathcal{G}$  is known as the Frobenius group  $F_{7,3}$ , which can be described as the additive group of the field  $\mathbb{F}_7$  extended with the multiplications by a nonzero square. It is the automorphism group of  $R$ , but also an

automorphism group of the Fano plane. Indeed,  $2R+I$ , and also  $J-2R-2I$  are incidence matrices of the Fano plane.

First we determine the possible columns of  $V$ . This means that we have to find the vectors  $\mathbf{v} \in \{0,1\}^7$  for which  $R^\top \mathbf{v}$  is again a  $(0,1)$  vector.

**Lemma 11.** *Let  $\mathbf{v} \in \{0,1\}^7$ . With  $R$  and  $P_1$  as above,  $R^\top \mathbf{v} \in \{0,1\}^7$  if and only if the vector  $\mathbf{v}$  or the complement  $\mathbf{1} - \mathbf{v}$  is equal to  $\mathbf{0}$ , or  $P_1^i[1101000]^\top$  for some  $i \in \{0, \dots, 6\}$ . If  $\mathbf{v} = P_1^i[1101000]^\top$ , or  $P_1^i[0010111]^\top$ , then  $R^\top \mathbf{v} = P_1^i[0010110]^\top$ , or  $P_1^i[1101001]^\top$ , respectively.*

*Proof.* This follows by straightforward verification. Using the above mentioned automorphisms of  $R$ , and the fact that  $R^\top(\mathbf{1} - \mathbf{v}) = \mathbf{1} - R^\top \mathbf{v}$ , there are just a few cases to be checked.  $\square$

Next we determine the set  $\mathcal{B}$  of adjacency matrices  $B$  of order 7, that have the property that  $B' = R^\top B R$  is a  $(0,1)$  matrix again. In the determination and description of  $\mathcal{B}$  we use that  $\mathcal{B}$  is invariant under the action of  $\mathcal{G}$ , and under complementation. More precisely, if  $B \in \mathcal{B}$ , then so is  $J - B - I$ , and  $P^\top B P$  for  $P \in G$ . Moreover,  $(J - B - I)' = R^\top(J - B - I)R = J - B' - I$  and  $(P^\top B P)' = R^\top P^\top B P R = P'^\top B' P$ .

**Lemma 12.** *Let  $B$  be an adjacency matrix of order seven. With  $R$ ,  $P_1$  and  $P_2$  as above, the matrix  $B' = R^\top B R$  is again an adjacency matrix if and only if  $B$  can be obtained from one of the following  $B_0 \dots B_{11}$  by complementation and/or a permutation of rows and columns generated by  $P_1$  and  $P_2$ .*

$$\begin{aligned}
B_0 = O, B_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \\
B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}, B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \\
B_6 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, B_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$B_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, B_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, B_{11} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The switched matrices  $B'_i = R^\top B_i R$  satisfy  $B'_0 = B_0$ ,  $B'_1 = B_1$ ,  $B'_i = Z_7 B_i Z_7$  for  $i = 2, \dots, 5$ ,  $B'_6 = Z_7 B_9 Z_7$ ,  $B'_9 = Z_7 B_6 Z_7$ ,  $B'_{10} = Z_7 B_2 Z_7$ , and

$$B'_7 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B'_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, B'_{11} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix},$$

where  $Z_7$  is the reverse identity matrix of order 7, that is,  $(Z_7)_{i,j} = 1$  if  $i + j = 7$ , and 0 otherwise.

Again the proof goes by straightforward verification. Observe that  $B_0$  to  $B_{11}$  are all nonisomorphic, and together with the complements this gives 24 nonisomorphic graphs for which the seven vertex switching works. Out of these graphs  $B_0$  and its complement are the only ones invariant under the group  $\mathcal{G}$ . Of the remaining cases  $B_1$  and its complement are invariant under the cyclic permutation  $P_1$ , and  $B_5, \dots, B_9$  and their complements are invariant under  $P_2$ . So in total there are 288 adjacency matrices  $B$  of order 7 for which  $B' = R^\top B R$  is again an adjacency matrix. For the six vertex switching we observed that  $B'_i$  is isomorphic with  $B_i$  in all cases. This is not true anymore for the seven vertex switching. Indeed,  $B'_i$  is nonisomorphic (and hence semi-isomorphic) to  $B_i$  for  $i = 6, \dots, 10$ . It is not difficult to see that these semi-isomorphic pairs can also be made by GM switching with respect to four vertices. However, the following example on eight vertices gives semi-isomorphic graphs that can be made by the seven vertex switching described above, but not by GM switching.

$$A = \left[ \begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right], A' = \left[ \begin{array}{cccccccc|c} 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right].$$

### 3.6 Eight vertex switching

In this section we consider the case that  $Q$  has one nontrivial indecomposable block  $R$  of order 8. Theorem 7 gives two nonequivalent possibilities for  $R$ , being:

$$R_1 = \frac{1}{2} \begin{bmatrix} J & O & O & Y \\ Y & J & O & O \\ O & Y & J & O \\ O & O & Y & J \end{bmatrix}, \text{ and } R_2 = \frac{1}{2} \begin{bmatrix} -I & I & I & I \\ I & -Z & I & Z \\ I & Z & -Z & I \\ I & I & Z & -Z \end{bmatrix},$$

with  $I, J, O, Y = 2I - J$ , and  $Z = J - I$  of order 2. We had hoped to find a general description of matrices  $B$  for which  $R^\top B R$  is a  $(0, 1)$  matrix again, when  $R$  has the form of Case (ii) in Theorem 7, but failed. Already for the above matrix  $R_1$  of order 8, we found 3584 such matrices, and we were not able to discover a general structure. Also for  $R_2$  we found a large number (1504) of such matrices  $B$ , so we decided not to give a complete description of the switching conditions as we did in the previous sections for six and seven vertex switching. However, in the next section we will investigate semi-isomorphism for graphs on eight vertices. Therefore we also have to consider eight vertex switching with no additional vertices, that is,  $Q = R$ . In this case we only have to consider adjacency matrices  $B$  for which  $B' = R^\top B R$  is nonisomorphic with  $B$ . With the help of a computer we found the following:

**Lemma 13.** *There exist exactly 20 nonisomorphic graphs  $G_1, \dots, G_{20}$ , which have an adjacency matrix  $B_i$  for which  $B'_i = R_1^\top B_i R_1$  is the adjacency matrix of a graph nonisomorphic with  $G_i$  for  $i = 1, \dots, 20$ . The matrices  $B_1, \dots, B_{10}$  of  $G_1 \dots G_{10}$  are displayed in Table 3.1, and  $G_{11}, \dots, G_{20}$  are the complements of  $G_1, \dots, G_{10}$ .*

*There exist exactly 36 nonisomorphic graphs  $G_{21}, \dots, G_{56}$ , which have an adjacency matrix  $B_i$  for which  $B'_i = R_2^\top B_i R_2$  is the adjacency matrix of a graph nonisomorphic with  $G_i$  for  $i = 21, \dots, 56$ . The matrices  $B_{21}, \dots, B_{38}$  of  $G_{21} \dots G_{38}$  are displayed in Table 3.2, and  $G_{39}, \dots, G_{56}$  are the complements of  $G_{21}, \dots, G_{36}$ .*

### 3.7 Semi-isomorphic graphs with eight vertices

With the results of the previous sections, we were able to generate by computer all graphs on eight vertices for which the six vertex switching (with



00110110 00111011 00111010 00111000 11000011 11000001 11000000 11000010 01000000 11000000 10000000 00000000 11100000 10010000 00100000 10100000	00011010 00101001 00101010 00101010 01001100 11001100 10001100 00001100 11110000 11110000 00110000 00110000 11000000 01000000 00000000 10000000	00111010 00101011 00001010 00101000 10001111 11001110 10001100 00001110 11110000 11110000 00110000 00110000 11100000 10110000 00100000 10000000	00110110 00111011 00111010 00111000 11001111 11001101 11001100 11001110 01110000 11110000 10110000 00110000 11100000 10010000 00100000 10100000
00110100 00111100 00111000 00110000 11001110 11001001 11000010 11001010 01100000 10110000 10100000 10000000 00110000 00010000 00000000 00100000	00111011 00101111 00001011 00100011 10001110 11001010 10000010 00001010 11100000 10110000 00100000 10000000 11110000 11110000 11000000 11000000	00110111 00111111 00111011 00110011 11001110 11001001 11000010 11001010 01100000 10110000 10100000 10000000 11110000 11010000 11000000 11100000	00110011 00010011 00000011 00100011 10000110 01001010 10001010 10001010 00010000 00110000 00100000 00000000 11110000 11110000 11000000 11000000
00111111 00011111 00001111 00101111 10000110 01001010 10001010 10001010 11010000 11110000 11100000 11000000 11110000 11110000 11000000 11000000	00011010 00101001 00101010 00101010 01001111 11001111 10001111 00001111 11110000 11110000 00110000 00110000 11110000 01110000 00110000 10110000		

**Table 3.1** Nonisomorphic pairs  $B_i, B'_i = R_1^\top B_i R_1$  ( $i = 1, \dots, 10$ ) mentioned in Lemma 13.

two extra vertices), the seven vertex switching (with one extra vertex), or one of the eight vertex switching (with no extra vertex) applies and gives a nonisomorphic (and therefore semi-isomorphic) mate. In total we found 427 nonisomorphic graphs; 227 by six vertex switching, 144 by seven vertex switching, and 56 (see Lemma 13) by eight vertex switching.

For  $n \leq 11$ , Table 1 of [42] gives exact numbers of nonisomorphic graphs on  $n$  vertices for which there exist an  $\mathbb{R}$ -cospectral mate (that is, the graph is not determined by the generalized spectrum); the column carries the name  $A \& \overline{A}$ . The table also presents the number of graphs for which a nonisomorphic cospectral mate can be obtained by GM switching (the name of the column is GM). If  $n \leq 8$ , only GM switching with respect to four vertices can

00001010 00111100 00100110 00010000 01011010 10011001 00101100 11100000 10110000 10100010 01010000 10000001 11100000 00001000 00000000 00100100	01100011 00111000 10100110 00010000 11011010 10011101 00101100 11100000 00110000 10100110 01010000 00101011 11100000 00001100 10000000 00100100	01001100 00011100 10101010 00000000 01001001 00010011 00001010 10101010 11110000 10010001 10000000 10000010 01010000 00110100 00100000 00101000	01110101 00001000 10001100 00101100 10010011 01011101 10100110 00100011 01000000 11100100 11010000 01101010 00110000 00010100 10100000 00110000
00011111 00101000 00000100 00111010 00010010 11011001 10101100 01100001 10010110 11100010 11011000 00000000 10101000 01001001 10000000 00110010	00111001 00101001 00010010 00110010 10010010 11011000 11101100 01101001 10010110 10110110 00011000 00001000 01101000 01001000 10000000 10010000	01011111 00101000 10010010 00110010 00010010 11011001 11101100 01101001 10010110 10110111 10011000 00001000 11101000 01001001 10000000 00111010	00001010 00111101 00101110 00010000 01010010 10011001 00101100 11100001 11010110 10100010 01011000 10000000 11101000 00001001 00000000 10110010
00111111 00101010 00000100 00111010 10001011 11011011 10001100 01100001 10110110 11100010 11011000 00000010 10101000 11101100 10100000 00110000	00001111 00111000 00100100 00011010 01011000 10010011 00101001 11100100 10110110 11000010 11001000 00010000 10001000 01101001 10010000 00100010	00101001 00111001 00110010 00010010 11011000 10010010 01101001 11101100 10110110 10010110 00001000 00011000 01001000 01101000 10010000 10000000	01000011 00110001 10111110 00010100 01010100 10010101 01100101 11100101 01000110 00000111 01111000 01111000 11001000 00001001 10010000 10111010
00011001 00100001 00001101 00111100 00010100 11010000 10100101 01100111 11000110 01000010 01111000 01010000 00001000 00011001 11010000 10010010	01011001 00100001 10011011 00110100 00010100 11010000 11100101 01101111 11000110 00010111 00111000 01011000 01001000 00011001 11010000 10011010	00110111 00001010 00000100 00111010 10000011 01001101 10001100 01000001 00010011 11100010 11010000 00100010 10101000 11001100 10101000 00110000	00011110 00001110 00010010 00110010 00000011 01001001 11001100 01001001 10010011 10110011 10010000 10000000 11101000 11001000 00101000 00111000
00111101 00001010 00000001 00111010 10001001 01000010 10001001 01001110 10110011 11010010 10000000 00010010 00001000 11111100 11111000 00000000	00101000 00011111 00101011 00010000 11001001 00000010 00001001 11001110 11110011 10010010 00000000 10010010 01001000 10111100 01111000 10000000		

**Table 3.2** Nonisomorphic pairs  $B_i, B'_i = R_2^\top B_i R_2$  ( $i = 21, \dots, 38$ ) mentioned in Lemma 13.

give nonisomorphic mates. Therefore, nonisomorphic pairs related by GM switching must be semi-isomorphic when  $n \leq 8$ . For  $n \leq 6$ , all graphs are determined by their generalized spectrum. On seven vertices, there exist 1044 graphs. Out of these, 40 graphs are not determined by the generalized spectrum, but for each of these graphs there exist a semi-isomorphic mate by GM switching. Thus, every graph on seven vertices which is not determined by its generalized spectrum, is semi-isomorphic to some other graph. On eight vertices, there are 12346 nonisomorphic graphs. Out of these 1166 are not determined by their generalized spectrum, and for 1054 of these, an  $\mathbb{R}$ -cospectral mate can be obtained by GM switching. Ted Spence (private communication) generated the remaining 112 graphs, and we compared these with the 427 graphs, for which six, seven or eight vertex switching applies. Only 44 of the 112 graphs in Spence's list did not occur in our list of 427. These 44 graphs consist of 22 pairs of  $\mathbb{R}$ -cospectral graphs, which are not isomorphic or semi-isomorphic. Thus we have:

**Proposition 14.** *On eight vertices, there exist 22 pairs of nonisomorphic  $\mathbb{R}$ -cospectral graphs for which no graph is semi-isomorphic with another graph. These are the twelve pairs of graphs displayed in Table 3.3, together with their complements (the last two pairs of the table are self-complementary).*

According to Theorem 1, each of the 22 pairs of matrices from Proposition 14 are similar by a regular orthogonal matrix  $Q$ . For example for the first pair in Table 3.3 we find

$$Q = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 2 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & 2 & 0 & 0 \\ 2 & -1 & -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

which is a regular orthogonal matrix of level 3.

**Warning.** The 22 pairs of Proposition 14 are not the only  $\mathbb{R}$ -cospectral pairs that are not semi-isomorphic with each other. For example  $G_1$  and  $G_2$  from Example 8 have the same property, but the two graphs nor the complements do occur in Table 3.3. The reason is that both graphs have a nonisomorphic cospectral mate by GM switching, therefore they are both semi-isomorphic with another graph, but not with each other.

01110010 01110011 10000011 10000010 10000010 10000010 10001101 10001100 00010000 00010001 00010000 00010000 11100000 11100000 01010000 10001000	01111000 01111100 10110000 10110000 11000100 11001000 11000100 11000011 10000111 10100000 00111010 10000011 00001100 00010100 00001000 00010100	01111000 01111100 10110100 10111000 11001000 11000010 11000000 11000000 10100100 11000000 01001011 10000011 00000100 00100100 00000100 00000100	01111000 01111100 10110100 10110010 11001010 11001000 11000001 11000001 10100001 10100001 01000001 10000010 00100000 01000100 00011100 00011000
01111100 01111100 10110010 10111000 11000010 11010011 11000010 11100000 10000111 11000000 10001000 10000011 01111000 00100100 00001000 00100100	01111100 01111100 10110010 10110010 11001000 11001001 11000001 11000100 10100100 10100000 10001011 10010001 01000100 01000001 00010100 00100110	01111100 01111100 10110010 10110010 11001000 11001010 11000001 11000001 10100101 10100000 10001000 10000000 01000000 01100001 00011000 00010010	01111100 01111100 10110010 10110010 11001001 11001010 11000100 11001001 10100100 10110000 10011010 10000000 01000100 01100001 00100000 00010010
01111100 01111100 10110010 10111010 11001010 11000110 11000011 11000100 10100001 11000001 10000000 10110001 01110001 01100000 00011010 00001100	01111100 01111110 10110011 10110001 11001010 11001000 11000000 11000100 10100100 10100010 10001001 10010000 01100000 10001001 01000100 01000010	01111100 01111100 10110010 10110010 11001000 11001010 11000011 11000101 10100101 10100001 10001010 10010000 01010100 01100001 00011000 00011010	01111110 01111110 10111001 10111110 11010001 11011000 11100001 11100000 11000001 11100000 10000000 11000001 10000000 11000000 01111000 00000100

**Table 3.3** Pairs of  $\mathbb{R}$ -cospectral graphs mentioned in Proposition 14, not semi-isomorphic with another graph.

Lemmas 12 and 13 show graphs which are semi-isomorphic by an indecomposable matrix  $Q$ . However, the computer investigations revealed that in all these cases there is also an decomposable  $Q$  that establishes the semi-isomorphism. We know of no pair of graphs which are semi-isomorphic only by an indecomposable regular orthogonal matrix of level 2.

### 3.8 Semi-isomorphism with respect to larger subgraphs

Lemma 9 can be generalized to all matrices  $R$  of type (ii) in Theorem 7. Equivalently, it means that we determine the possible columns of  $V$ , i.e., we find the vectors  $\mathbf{v} \in \{0, 1\}^b$  for which  $R^\top \mathbf{v}$  is again a  $(0, 1)$  vector.

**Lemma 15.** *Let  $\mathbf{v} \in \{0, 1\}^b$ , and let  $R$  be a  $b \times b$  matrix of type (ii) in Theorem 7. Then,  $R^\top \mathbf{v} \in \{0, 1\}^b$  if and only if the numbers of ones in each class of the partition is even, or the number of ones in each class of the partition is odd. In the first case  $R^\top \mathbf{v} = \mathbf{v}$ . For the second case, let  $A = 01$  and  $B = 10$  be the two possible classes of the partition and  $\mathbf{v} = [C_1 C_2 C_3 \dots C_{\frac{b}{2}}]^\top$  with  $C_i \in \{A, B\}$  for  $i = 1, \dots, \frac{b}{2}$ . Then,  $R^\top \mathbf{v} \in \{A, B\}^{\frac{b}{2}}$ , and in particular,  $R^\top \mathbf{v} = [C_2 C_3 \dots C_{\frac{b}{2}} C_1]^\top$ .*

*Proof.* With  $\mathbf{v} = [v_1 \dots v_b]^\top$  we have

$$\mathbf{0} = 2R^\top \mathbf{v} = \begin{bmatrix} v_1 + v_2 + v_3 - v_4 \\ v_1 + v_2 - v_3 + v_4 \\ v_3 + v_4 + v_5 - v_6 \\ v_3 + v_4 - v_5 + v_6 \\ v_1 - v_2 + v_5 + v_6 \\ -v_1 + v_2 + v_5 + v_6 \\ \dots \\ v_{b-3} - v_{b-2} + v_{b-1} + v_b \\ -v_{b-3} + v_{b-2} + v_{b-1} + v_b \\ v_1 - v_2 + v_{b-1} + v_b \\ -v_1 + v_2 + v_{b-1} + v_b \end{bmatrix} = \begin{bmatrix} v_{1,2} + v_{3,4} \\ v_{1,2} + v_{3,4} \\ v_{3,4} + v_{5,6} \\ v_{3,4} + v_{5,6} \\ v_{1,2} + v_{5,6} \\ v_{1,2} + v_{5,6} \\ \dots \\ v_{b-3,b-2} + v_{b-1,b} \\ v_{b-3,b-2} + v_{b-1,b} \\ v_{1,2} + v_{b-1,b} \\ v_{1,2} + v_{b-1,b} \end{bmatrix} \pmod{2},$$

where  $v_{i,i+1} = v_i + v_{i+1}$  for  $i = 1, 3, 5, \dots, b-1$ . It follows that  $R^\top \mathbf{v}$  is a  $(0, 1)$  vector if and only if  $v_{1,2} = v_{3,4} = v_{5,6} = \dots = v_{b-1,b} \pmod{2}$ . The second part of the lemma follows by straightforward verification.  $\square$

# Godsil-McKay Switching and Isomorphism

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As we have seen, Godsil-McKay switching is an operation on graphs that does not change the spectrum of the adjacency matrix. Usually (but not always) the obtained graph is nonisomorphic with the original graph. In this chapter we present a straightforward sufficient condition for being isomorphic after switching, and give examples which show that this condition is not necessary. For some graph products we obtain sufficient conditions for being nonisomorphic after switching. As an example we find that the tensor product of the  $\ell \times m$  grid ( $\ell > m \geq 2$ ) and a graph with at least one vertex of degree two is not determined by its adjacency spectrum.

## 4.1 Godsil-McKay switching

Godsil-McKay switching (see Lemma 2) is also used in this chapter. We shall call the considered partition in the GM switching a *(Godsil-McKay) switching partition*. In many applications  $\ell = 1$ . Then the condition of GM switching requires that  $X = X_1$  induces a regular subgraph of  $G$ , and that each vertex in  $Y$  has  $0$ ,  $\frac{1}{2}|X|$  or  $|X|$  neighbors in  $X$ . Such a set  $X$  will be called a *(Godsil-McKay) switching set*. In this section we look for conditions (necessary and/or sufficient) on a switching set under which  $G$  and  $G'$  are isomorphic.

Let  $G$  be a graph with adjacency matrix  $A$  and switching set  $X$ . Let  $B$  be the submatrix of  $A$  corresponding to  $X$ . Then

$$A = \begin{bmatrix} B & M \\ M^\top & C \end{bmatrix}, \quad \text{with } M = \begin{bmatrix} N & J & O \end{bmatrix},$$

where  $BJ = kJ$  for some  $k \in \{0, \dots, |X| - 1\}$ , and  $N^\top J = \frac{1}{2}|X|J$ . Note that not every (but at least one) type of block  $N$ ,  $J$  or  $O$  needs to be present. Let  $G'$  be the graph with adjacency matrix  $A'$  obtained by Godsil-McKay switching with respect to  $X$  in  $G$ . Then

$$A' = \begin{bmatrix} B & M' \\ M'^\top & C \end{bmatrix}, \quad \text{with } M' = \begin{bmatrix} J - N & J & O \end{bmatrix}.$$

With the above notation, the following proposition is straightforward.

**Proposition 16.** *If there exist permutation matrices  $P$  and  $Q$  such that  $PBP^\top = B$ ,  $PMQ^\top = M'$  and  $QCQ^\top = C$ , then  $G$  and  $G'$  are isomorphic.*

*Proof.* Two graphs  $G$  and  $G'$ , with adjacency matrices  $A$  and  $A'$ , respectively, are isomorphic if and only if there exist a permutation matrix  $R$  such that  $A' = RAR^\top$ . Take  $R = \begin{bmatrix} P & O \\ O & Q \end{bmatrix}$ , then

$$A' = RAR^\top = \begin{bmatrix} PBP^\top & PMQ^\top \\ QM^\top P^\top & QCQ^\top \end{bmatrix}.$$

□

Any pair of vertices in  $G$  is a switching set, but such a set always satisfies the above proposition, so switching produces isomorphic graphs. However, if  $|X| \geq 4$  then Proposition 16 is not automatically satisfied and Godsil-McKay switching usually (but not always) produces nonisomorphic graphs. To prove that  $G$  and  $G'$  are nonisomorphic it would help if the condition of Proposition 16 would also be necessary for isomorphism. This however is not true! The isomorphism described in the proposition fixes the switching set  $X$  (setwise). We shall see examples in the next section where  $G$  and  $G'$  are isomorphic, but no isomorphism fixes  $X$ . Because of these examples it will be hard to find useful conditions for isomorphism that are necessary and sufficient. Therefore we only present some easy sufficient conditions for being nonisomorphic after Godsil-McKay switching. Let  $\lambda_G(x, y)$  denote the number of common neighbors of two vertices  $x$  and  $y$  in  $G$ . It is clear that if the multiset of degrees (i.e.  $\{\lambda_G(x, x) \mid x \in V(G)\}$ ), or the multiset  $\{\lambda_G(x, y) \mid x, y \in V(G)\}$  changes after switching, then  $G$  and  $G'$  are nonisomorphic. But we can be a bit more precise:

**Lemma 17.** *The following conditions are sufficient for  $G$  and  $G'$  being nonisomorphic.*

- i* The multiset of degrees (in  $G$ ) of the vertices in  $X$  changes after switching.
- ii* The multiset  $\Lambda_G = \{\lambda_G(x, y) \mid x \in X, y \in V(G)\}$  changes after switching.
- iii* The vertices of  $X$  all have the same degree, and the multiset  $\bar{\Lambda}_G = \{\lambda_G(x, y) \mid x \in X, y \in Y\}$  changes after switching.

**Proof.** (i) Clearly the degrees in  $Y$  don't change by the switching, so the multiset of degrees of  $G$  changes whenever the degrees in  $X$  change. (ii) The multiset  $\{\lambda_G(x, y) \mid x, y \in Y\}$  is not changed after switching, therefore  $\{\lambda_G(x, y) \mid x, y \in V(G)\}$  changes if  $\Lambda_G(G)$  changes. (iii) If the vertices in  $X$  have the same degree, then switching does not change  $\{\lambda_G(x, y) \mid x, y \in X\}$ .  $\square$

Suppose not all vertices in  $X$  have the same degree. Then in most cases the set of degrees changes, and hence we get a nonisomorphic graph after switching. In particular this is always the case if  $|X| = 4$ .

The conditions of Lemma 17 are not necessary for being nonisomorphic. There are several examples of Godsil-McKay switching in a strongly regular graph  $G$  that gives a nonisomorphic graph  $G'$  (the smallest example is the  $4 \times 4$  grid with a coclique  $X$  of size 4). However,  $G'$  is also strongly regular with the same parameters as  $G$  (since this property follows from the spectrum), and therefore  $\Lambda_G = \Lambda_{G'}$  and  $\bar{\Lambda}_G = \bar{\Lambda}_{G'}$ .

## 4.2 No isomorphism fixes the switching set

In this section we give examples of graphs  $G$  with a switching set  $X$  for which the graphs  $G'$  obtained by Godsil-McKay switching are isomorphic with  $G$ , but where no isomorphism fixes  $X$ .

### 4.2.1 Regular tournaments

A  $(0, 1)$ -matrix  $T$  is a *tournament matrix* if  $T + T^\top = J - I$ , and  $T$  is *regular* if all row (and column) sums are equal. If  $T$  has order  $m$ , then this row sum is  $(m - 1)/2$ , so  $m$  is odd.

**Proposition 18.** *Let  $T$  be a regular tournament matrix of order  $m > 1$ , and put  $N = T \otimes J_2 + I_{2m}$ . Consider a regular graph  $H$  of order  $2m$  with vertex*



set  $X$  and automorphism  $\rho$  that is a fixed-point-free involution, where the orbits of the full automorphism group of  $H$  are the orbits of  $\rho$ . Let  $H$  have adjacency matrix  $B$ , indexed such that  $\rho$  is represented by the permutation matrix  $R = I_m \otimes (J_2 - I_2)$ . Construct a graph  $G$  on the union of two copies  $X_1, X_2$  of  $X$ , with adjacency matrix

$$A = \begin{bmatrix} B & N \\ N^\top & B \end{bmatrix}.$$

Then  $G$  has Godsil-McKay switching set  $X_1$ , and the switched graph  $G'$  is isomorphic with  $G$ , whilst there is no isomorphism that fixes  $X_1$ .

**Proof.** We have  $RN = J - N^\top$  and  $B = RBR^\top$ , and therefore  $A' = QAQ^\top$ , where

$$Q = \begin{bmatrix} O & I \\ R & O \end{bmatrix}.$$

Thus  $G$  is isomorphic with  $G'$ . Suppose there is an isomorphism between  $G$  and  $G'$  that fixes the set  $X_1$  (and hence also  $X_2$ ). Then the isomorphism acts as an automorphism on the subgraphs induced by  $X_1$  and  $X_2$ , and hence fixes the orbits of  $\rho$  on both copies of  $X$ . Since  $m > 1$  this is impossible.  $\square$

Regular tournament matrices are easily constructed for every odd order  $m$ . If  $E$  is the adjacency matrix of an asymmetric regular graph (asymmetric means that the full automorphism group is trivial), then  $E \otimes J_2$  represents a graph whose automorphism group satisfies the condition of the proposition. An asymmetric regular graph exists for every order at least 10 (see [10]), but also for  $m = 5, 7$  and  $9$  graphs with the required property do exist. For example when  $m = 5$  we can take

$$B = \begin{bmatrix} Z & O & Z & O & J \\ O & Z & J & Z & O \\ Z & J & O & Z & O \\ O & Z & Z & O & J \\ J & O & O & J & O \end{bmatrix}, \text{ and } N = \begin{bmatrix} I & J & J & O & O \\ O & I & J & J & O \\ O & O & I & J & J \\ J & O & O & I & J \\ J & J & O & O & I \end{bmatrix},$$

where  $J = J_2$ ,  $I = I_2$  and  $Z = J_2 - I_2$ . So the construction works for every order  $4m$  with  $m$  odd and at least 5. The smallest size of the switching set is 10. Since in many applications the size of the switching set is 4, the question rises whether in this special case the sufficient condition for isomorphism of Proposition 16 could be necessary. Unfortunately this is again false, as is illustrated by the next example.

### 4.2.2 A switching set of size four

Let  $G$  be the bipartite graph on  $12 + 6 = 18$  vertices, where one part of the bipartition is  $\{a, b, c, d, a', b', c', d', a'', b'', c'', d''\}$ , the other is  $\{u_i \mid i \in \mathbb{Z}/6\mathbb{Z}\}$ , and adjacencies are

$$\begin{aligned} u_0 &\sim a, b, a', c', a'', d'' \\ u_1 &\sim b, c, a', b', a'', c'' \\ u_2 &\sim b, d, b', c', a'', b'' \\ u_3 &\sim c, d, b', d', b'', c'' \\ u_4 &\sim a, d, c', d', b'', d'' \\ u_5 &\sim a, c, a', d', c'', d'' \end{aligned}$$

Let the switching set be  $X = \{a, b, c, d\}$ . Then we have an isomorphism between  $G$  and the switched graph  $G'$ .  $\phi : G \rightarrow G'$  given by  $\phi(x) = x'$ ,  $\phi(x') = x''$ ,  $\phi(x'') = x$  for  $x = a, b, c, d$ , and  $\phi(u_i) = u_{i+1}$  for  $i \in \mathbb{Z}/6\mathbb{Z}$ . We would like to show that there is no isomorphism fixing  $X$  (but there is). Put  $X' = \{a', b', c', d'\}$  and  $X'' = \{a'', b'', c'', d''\}$  and  $U = \{u_i \mid i \in \mathbb{Z}/6\mathbb{Z}\}$ . The graphs  $G$  and  $G'$  are bipartite and connected, so any isomorphism  $\psi$  fixing  $X$  must also fix  $X' \cup X''$  and  $U$ . The triples  $ijk$  such that  $u_i, u_j, u_k$  have a common neighbor in  $X$  are 045, 012, 135, 234, and after switching 123, 345, 024, 015, so  $\psi$  must send the former triples to the latter. The former triples are precisely the triples with a common neighbor in  $X''$ , the latter precisely those with a common neighbor in  $X'$ . So  $\psi$  must interchange  $X'$  and  $X''$ . As it turns out, there is such a  $\psi$ , and we need to enlarge our graph to destroy this unwanted isomorphism.

We can turn the 18-vertex non-example into a 21-vertex almost-example by adding three vertices  $X$ ,  $X'$  and  $X''$ , corresponding to the sets with the same names, adjacent to their elements (thus:  $X \sim a, b, c, d$ , etc.), and three directed edges  $X \rightarrow X'$ ,  $X' \rightarrow X''$ , and  $X'' \rightarrow X$ . This gets rid of automorphisms  $\psi$  preserving  $X$ , but the example is directed. However, Frucht [31] showed that every finite group is the full group of automorphisms of some finite undirected graph. In particular we can find a graph with full group  $C_3$ , the cyclic group of order 3, and use that instead of the directed edges. This yields an actual example. Let us give an explicit example on 9 vertices ([54]). Take 9 vertices  $x_i$  with  $x$  one of  $a, b, c$  and  $i \in \mathbb{Z}/3\mathbb{Z}$ . The 15 edges are  $a_i b_i$ ,  $a_i c_{i-1}$ ,  $b_i c_i$ ,  $b_i b_{i+1}$ ,  $b_i c_{i-1}$ . This yields a graph with  $C_3$  as full group of automorphisms. Identify the vertices  $X, X', X''$  of the 21-vertex

almost-example with the vertices  $a_0$ ,  $a_1$  and  $a_2$  of this gadget (and remove the directed edges) to obtain a 27-vertex example as claimed.

### 4.3 Graph products

Consider graphs  $G$  and  $H$  with adjacency matrices  $A$  and  $E$ , respectively. We recall that the *tensor product* of  $H$  and  $G$ , denoted by  $H \times G$  is the graph with adjacency matrix  $E \otimes A$ . We will also consider another product, which we will call the *strengthened tensor product*, defined by its adjacency matrix  $(E + I) \otimes A$ , and denoted by  $H \bowtie G$ . Notice that the strengthened tensor product  $H \bowtie G$  can be interpreted as a tensor product  $\overline{H} \times G$  where  $\overline{H}$  is obtained from  $H$  by adding a loop at every vertex.

Let  $X$  be a switching set in  $G$  and suppose that one of the conditions of Lemma 17 is satisfied, so that  $G$  is nonisomorphic and cospectral with  $G'$ . Then it is easily checked that also the products  $G \times H$  and  $G \bowtie H$  are nonisomorphic and cospectral with  $G' \times H$  and  $G' \bowtie H$ , respectively. Indeed, nonisomorphism easily follows because  $\lambda_{H \times G}((i, x), (j, y)) = \lambda_H(i, j)\lambda_G(x, y)$  and  $\lambda_{H \bowtie G}((i, x), (j, y)) = \lambda_{\overline{H}}(i, j)\lambda_G(x, y)$ , therefore also the multisets  $\{\lambda_{H \times G}((i, x), (j, y)) \mid i, j \in V(H), x, y \in V(G)\}$  and  $\{\lambda_{H \bowtie G}((i, x), (j, y)) \mid i, j \in V(H), x, y \in V(G)\}$  are changed after switching (assuming that  $H$ , resp.  $\overline{H}$ , has at least one edge). Cospectrality follows from basic properties of tensor products of matrices, but also from the observation that in both products the sets  $\{X_i = \{i\} \times X\}$ , with  $i \in V(H)$ , together with the set  $Y$  of remaining vertices is a switching partition.

If none of the conditions of Lemma 17 is satisfied, so that it is conceivable that  $G$  is isomorphic with  $G'$ , then under some easy conditions there exist switching sets in  $H \times G$  and  $H \bowtie G$  that lead to nonisomorphic graphs. For the formulation of the result we will use the notation of Section 2, and the notion of a pair of *complementary rows* in a  $(0, 1)$  matrix, which simply means that the sum of the two rows is equal to the all-one row. Recall that given two vertices  $x$  and  $y$  of  $G$ , we defined the multiset  $\overline{\Lambda}_G$  as  $\overline{\Lambda}_G = \{\lambda_G(x, y) \mid x \in X, y \in Y\}$ .

**Theorem 19.** *Let  $G$  be a graph with a Godsil-McKay switching set  $X$ , such that the vertices of  $X$  have the same degree, and suppose that  $\overline{\Lambda}_G = \overline{\Lambda}_{G'}$ . Furthermore suppose that either  $X$  is a coclique (i.e.  $B = O$ ),  $N$  has at least two columns and no pair of complementary rows, or that  $B$  has row sums  $\frac{1}{2}|X|$  and no pair of rows of  $[B \ N]$  is complementary. Let  $H$  be a graph*

and let  $i$  be a vertex of  $H$ . Then the subset  $\{i\} \times X$  of  $V(H) \times V(G)$  is a switching set in  $H \times G$  as well as in  $H \bowtie G$ , and Godsil-McKay switching gives nonisomorphic cospectral graphs, provided that  $i$  has degree at least one in case of the strengthened tensor product and  $i$  is adjacent to a vertex of degree at least two in case of the tensor product.

**Proof.** It is easily checked that for both graph products, the set  $\{i\} \times X$  is a switching set. We'll apply Lemma 17(iii) and prove that the multisets  $\bar{\Lambda}_{H \times G}$  and  $\bar{\Lambda}_{H \bowtie G}$  change after switching.

First observe that the Kronecker products  $E \otimes A$  and  $(E + I) \otimes A$  consist of blocks matrices equal to  $A$  or  $O$ . After switching the blocks equal to  $A$  in the block row and block column corresponding to  $i$  change, but the other blocks remain the same. For the strengthened tensor product, the diagonal block corresponding to  $i$  becomes the switched matrix  $A'$ . For both graph products the off-diagonal nonzero blocks in block row  $i$  become  $A''$ , which is obtained from  $A$  by switching with respect to the rows corresponding to  $X$ . Note that we can obtain  $A''$  also from  $A'$  by switching with respect to the columns corresponding to  $X$ . From this it follows that  $A''A''^\top = A'A'^\top$ .

For convenience we restrict to the tensor product in the remainder of the proof; the proof for the strengthened tensor product goes analogously. The multiset  $\bar{\Lambda}_{H \times G}$  consists of the values  $\lambda_{H \times G}((i, x), (j, y))$  where  $(i, x) \in \{i\} \times X$  and  $(j, y) \notin \{i\} \times X$ . We distinguish three cases.

Case (i):  $i = j$ . We have

$$\{\lambda_{H \times G}((i, x), (i, y)) \mid x \in X, y \in Y\} = \{\lambda_H(i, i)\lambda_G(x, y) \mid x \in X, y \in Y\},$$

and  $A'A'^\top = A''A''^\top$  implies that

$$\{\lambda_{(H \times G)'}((i, x), (i, y)) \mid x \in X, y \in Y\} = \{\lambda_H(i, i)\lambda_{G'}(x, y) \mid x \in X, y \in Y\}.$$

By assumption the multiset  $\bar{\Lambda}_G$  does not change after switching and therefore the multiset  $\{\lambda_{H \times G}((i, x), (i, y)) \mid x \in X, y \in Y\}$  is also invariant under switching.

Case (ii):  $i \neq j$  and  $y \in Y$ . For each  $j \neq i$  we have

$$\{\lambda_{(H \times G)'}((i, x), (j, y)) \mid x \in X, y \in Y\} = \{\lambda_H(i, j)\lambda_{G'}(x, y) \mid x \in X, y \in Y\} =$$

$$\{\lambda_H(i, j)\lambda_G(x, y) \mid x \in X, y \in Y\} = \{\lambda_{H \times G}((i, x), (j, y)) \mid x \in X, y \in Y\}.$$

Case (iii):  $i \neq j$  and  $x, y \in X$ . Choose  $\hat{j} \neq i$  such that  $\lambda_H(i, \hat{j})$  is maximal. It follows that  $\lambda_H(i, \hat{j}) > 0$  because  $i$  has a neighbor of degree at least two.

(Note that for the strengthened tensor product it suffices that the degree of  $i$  is at least 1.) We have  $\lambda_{H \times G}((i, x), (\hat{j}, x)) = \lambda_H(i, \hat{j})\lambda_G(x, x)$ . After switching we get  $\lambda_{(H \times G)'}((i, x), (\hat{j}, x)) = \lambda_H(i, \hat{j})\mu(x)$ , where  $\mu(x)$  is the number of neighbors of  $x$  that remain a neighbor after switching. Clearly  $\mu(x) < \lambda_G(x, x)$ , hence

$$\lambda_{(H \times G)'}((i, x), (\hat{j}, x)) < \lambda_{H \times G}((i, x), (\hat{j}, x)).$$

For  $y \neq x$  we get  $\lambda_{(H \times G)'}((i, x), (\hat{j}, y)) = \lambda_H(i, \hat{j})\lambda_{G'}(x, y)$ . Because the matrices  $N$  or  $[B \ N]$  which are switched to their complements have no complementary pair of rows, it follows that  $\lambda_{G'}(x, y) < \lambda_G(x, x)$ . Hence we have

$$\lambda_{(H \times G)'}((i, x), (\hat{j}, y)) < \lambda_H(i, \hat{j})\lambda_G(x, x) = \lambda_{H \times G}((i, x), (\hat{j}, x)).$$

This implies that the number  $\lambda_{H \times G}((i, x), (\hat{j}, x))$  disappears at least once from the multiset  $\bar{\Lambda}_{H \times G}$  after switching.  $\square$

In view of the previous section it seems relevant to remark that the proof of the above theorem would have been much simpler if we could have used that there exists an isomorphism that fixes the switching set.

The  $\ell \times m$  grid (or lattice graph  $L(\ell, m)$ ) is the line graph of the complete bipartite graph  $K_{\ell, m}$  (we assume  $\ell \geq m$ ). If  $(\ell, m) \neq (4, 4)$ , or  $(6, 3)$ , then  $L(\ell, m)$  is determined by its spectrum. If  $\ell \geq 3, m \geq 2$  a 4-cycle in the grid is a switching set that satisfies the hypothesis of Theorem 19. Therefore the tensor product of  $L(\ell, m)$  ( $\ell \geq 3, m \geq 2$ ) and a graph with at least one vertex of degree two is not determined by its adjacency spectrum.

The strengthened tensor product  $K_n \boxtimes G$  ( $n > 1$ ) is also known as a *coclique extension* of  $G$ . So the above theorem gives some easy conditions for a coclique extension to have nonisomorphic cospectral graphs. For example a coclique extension of the grid  $L(\ell, m)$  with  $\ell \geq 3, m \geq 2$ , is not determined by its spectrum.

Another example is the triangular graph  $T(m)$ , which is the line graph of  $K_m$ . If  $m \neq 8$  the spectrum determines  $T(m)$  and if  $m \geq 4$  a 4-cycle in  $T(m)$  satisfies the requirements of Theorem 19. Thus we can conclude that for  $m \geq 4$  a coclique extension of  $T(m)$  is not determined by its spectrum.

# Switched symplectic graphs and their 2-ranks

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It is well-known that if a graph  $G'$  has the same spectrum as a strongly regular graph  $G$ , then  $G'$  is also strongly regular with the same parameters as  $G$  (see for example [14]). Therefore Godsil-McKay switching provides a tool to construct new strongly regular graphs from known ones. However, there is no guarantee that the switched graph is nonisomorphic with the original graph. In this chapter we use the 2-rank of the adjacency matrix (rank of the adjacency matrix over  $\mathbb{F}_2$ ) to prove non-isomorphism after switching.

For  $\nu \geq 2$ , the symplectic graph over  $\mathbb{F}_2$ , denoted by  $Sp(2\nu, 2)$  and which will be defined in Section 5.1, is a strongly regular with parameters  $(2^{2\nu} - 1, 2^{2\nu-1}, 2^{2\nu-2}, 2^{2\nu-2})$ . The 2-rank of the adjacency matrix of  $Sp(2\nu, 2)$  equals  $2\nu$ , which is the smallest possible value. The symplectic graph is characterized by Peeters [52] as follows.

**Theorem 20.** *The symplectic graph  $Sp(2\nu, 2)$  is uniquely determined by its parameters and its 2-rank.*

When  $\nu = 1$  we have the complete graph  $K_3$ , and  $Sp(4, 2)$  is a strongly regular graph with parameters  $(15, 8, 4, 4)$ , which is known to be determined by its parameters [60]. For  $\nu \geq 3$  we find Godsil-McKay switching sets in  $Sp(2\nu, 2)$  and prove that the 2-rank increases after switching, which implies that the switched graph is nonisomorphic with the original graph.

It turns out that for  $\nu \geq 3$  the symplectic graph has many switching sets that remain switching sets after switching. Therefore it is interesting to find out what happens after several switchings. We investigated this by computer

for  $Sp(6, 2)$  and found 1826 new strongly regular graphs with parameters  $(63, 32, 16, 16)$ . For the 2-rank of these new graphs we found six different values.

The symplectic graphs will be defined below, and in Section 5.5 we give an alternative description by use of a well-known recursive construction of Hadamard matrices. We settle the behavior of the 2-ranks of this recursive construction, and apply it to the strongly regular graphs with various 2-ranks found by computer. As a result we find that for every  $\nu \geq 3$  there exist strongly regular graphs with the same parameters as  $Sp(2\nu, 2)$  for a number of distinct values for the 2-rank. Moreover, this number of different 2-ranks is nondecreasing and goes to infinity when  $\nu \rightarrow \infty$ .

## 5.1 The symplectic graphs over $\mathbb{F}_2$

Let  $\mathbb{F}_2^{2\nu}$  be the  $2\nu$ -dimensional vector space over  $\mathbb{F}_2$ , and let  $K = I_\nu \otimes (J_2 - I_2)$ , where  $I_\nu$  is the identity matrix of order  $\nu$ , and  $J_2$  denotes the all-ones matrix of order 2. The *symplectic graph*  $Sp(2\nu, 2)$  over  $\mathbb{F}_2$  is the graph whose vertices are the nonzero vectors of  $\mathbb{F}_2^{2\nu}$ , where two vertices  $x$  and  $y$  are adjacent whenever  $x^\top K y = 1$ . Equivalently,  $x = [x_1 \ \dots \ x_{2\nu}]^\top$  and  $y = [y_1 \ \dots \ y_{2\nu}]^\top$  are adjacent if

$$(x_1 y_2 + x_2 y_1) + (x_3 y_4 + x_4 y_3) + \dots + (x_{2\nu-1} y_{2\nu} + x_{2\nu} y_{2\nu-1}) = 1.$$

For  $\nu \geq 2$ , it is known (see for example [52]) that the symplectic graph  $Sp(2\nu, 2)$  is a strongly regular graph with parameters

$$(2^{2\nu} - 1, 2^{2\nu-1}, 2^{2\nu-2}, 2^{2\nu-2}),$$

and eigenvalues  $2^{2\nu-1}$ ,  $2^{\nu-1}$ ,  $-2^{\nu-1}$  with multiplicities 1,  $2^{2\nu-1} - 2^{\nu-1} - 1$ ,  $2^{2\nu-1} + 2^{\nu-1} - 1$ , respectively.

## 5.2 Godsil-McKay switching and its 2-rank behavior

In this section we will make use, once more, of the Godsil and McKay switching [33]. According to the notation of Lemma 2, in this section we consider  $\ell = 1$  and  $X = X_1$  as a (Godsil-McKay) switching set. Note that any vertex subset of  $G$  of size two satisfies the required conditions of the GM switching,

but in this case the switched graph  $G'$  is isomorphic with  $G$ . Therefore we assume that a switching set has at least four vertices.

Let  $A$  and  $A'$  be the adjacency matrices of  $G$  and  $G'$ , respectively, and assume that the first  $|X|$  rows (and columns) of  $A$  and  $A'$  correspond to the switching set  $X$  and the last  $h$  rows correspond to the vertices outside  $X$  with exactly  $\frac{1}{2}|X|$  neighbors in  $X$ . Then

$$A' = A + M \pmod{2}, \quad \text{where } M = \begin{bmatrix} O & O & J \\ O & O & O \\ J^\top & O & O \end{bmatrix},$$

and  $J$  is the  $|X| \times h$  all-ones matrix. Since  $2\text{-rank}(M) = 2$ , the 2-ranks of  $A$  and  $A'$  differ by at most 2. It is well-known that the 2-rank of any adjacency matrix is even (see [16]), thus we have the following result.

**Proposition 21.** *Suppose  $2\text{-rank}(A) = r$ , then  $r$  is even and  $2\text{-rank}(A') = r - 2, r$ , or  $r + 2$ .*

### 5.3 Switched symplectic graphs

For  $\nu \geq 3$ , we define the following vectors in  $\mathbb{F}_2^{2\nu}$ :

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ z \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ z \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ z \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ z \end{bmatrix},$$

where  $z$  is an arbitrary vector in  $\mathbb{F}_2^{2\nu-6}$ .

**Proposition 22.** *The set  $X = \{v_1, v_2, v_3, v_4\}$  is a Godsil-McKay switching set of  $Sp(2\nu, 2)$  for  $\nu \geq 3$ .*

*Proof.* Any two vertices from  $X$  are nonadjacent, so the subgraph of  $Sp(2\nu, 2)$  induced by  $X$  is a coclique, and therefore regular. Consider an arbitrary vertex  $y \notin X$ . Then

$$y^\top K v_1 + y^\top K v_2 + y^\top K v_3 + y^\top K v_4 = y^\top K (v_1 + v_2 + v_3 + v_4) = y^\top K \mathbf{0} = 0.$$

This implies that the number of edges between  $y$  and  $X$  is even, and therefore  $X$  is a switching set.  $\square$



Let  $G'$  be the graph obtained from  $G = Sp(2\nu, 2)$  by switching with respect to  $X$ . We shall now prove that  $G$  and  $G'$  are nonisomorphic.

**Theorem 23.** *For  $\nu \geq 3$ , the graph  $G'$  obtained from  $Sp(2\nu, 2)$  by switching with respect to the switching set  $X$  given above, is strongly regular with the same parameters as  $Sp(2\nu, 2)$ , but with 2-rank equal to  $2\nu + 2$ .*

*Proof.* Let  $A$  be the adjacency matrix of  $G = Sp(2\nu, 2)$ , and assume that the first four rows and columns correspond to  $X$ . Then  $\text{2-rank}(A) = 2\nu$  and  $A$  has  $2^{2\nu} - 1$  rows. This implies that, over  $\mathbb{F}_2$ , every possible nonzero linear combination of a basis of the row space of  $A$  is a row of  $A$ . Therefore the sum (mod 2) of any two rows of  $A$  is again a row of  $A$ . Let  $r_1$  and  $r_2$  be rows of  $A$  corresponding to the vertices  $v_5 = [100000z^\top]^\top$  and  $v_6 = [001000z^\top]^\top$ , respectively. Then  $r_1$  starts with 0011 and  $r_2$  starts with 0101. It follows that  $r_7 = r_5 + r_6$  is also a row of  $A$  starting with 0110. After switching only the first four entries of  $r_5$ ,  $r_6$  and  $r_7$  change and become 1100, 1010 and 1001, respectively. Let  $r'_i$  denote the switched version of  $r_i$  ( $i = 5, 6$  or  $7$ ). Then  $v = r'_5 + r'_6 + r'_7 = 11110 \dots 0$ . So  $v$  is in the row space of the switched matrix  $A'$ , but it is not a row of  $A'$ . So  $G'$  is not isomorphic to  $G$ , and by Theorem 20 and Proposition 21 the 2-rank of  $A'$  equals  $2\nu + 2$ .  $\square$

The switching set  $X$  given above, is not the only one. There are many more and many remain a switching set after switching with respect to  $X$ . Therefore we can apply switching several times. However it is not true in general that a second switching increases the 2-rank again, and it looks difficult to make a general statement like in the above theorem. Instead we investigated the repeated switching by computer for the case  $\nu = 3$ .

It is also worthwhile to mention that in [39] upper bounds for the 2-rank of strongly regular graphs in terms of the eigenvalues are given. For graphs with the same parameters of  $Sp(2\nu, 2)$ , the 2-rank of its adjacency matrix  $A$  is bounded from above by  $2^{2\nu-1} - 2^{\nu-1}$ . But in fact, it can be improved, since the spectrum implies that the matrix

$$E = (2^{2\nu} - 1)(A + (2^{\nu-1})I) - (2^{2\nu-1} + 2^{\nu-1})J$$

has real rank equal to  $2^{2\nu-1} - 2^{\nu-1} - 1$ , and therefore the 2-rank of  $E$  is at most  $2^{2\nu-1} - 2^{\nu-1} - 1$ . Since  $A \equiv E \pmod{2}$  and the 2-rank of  $A$  must be even, it follows that the 2-rank is upper bounded by  $2^{2\nu-1} - 2^{\nu-1} - 2$ .

## 5.4 Repeated switching in $Sp(6, 2)$

In this section we show that Godsil-McKay switching generates a significant number of nonisomorphic graphs with the same parameters as the symplectic graph  $Sp(6, 2)$ . By computer we search for all switching sets of size 4 in  $Sp(6, 2)$ . We switch and compute the 2-rank. With the firstly encountered graph for which the 2-rank has increased, we repeat the procedure. We stop if the 2-rank cannot be increased. By this procedure we obtained 1827 nonisomorphic graphs with the parameters of  $Sp(6, 2)$ . The possible 2-ranks are: 6, 8, 10, 12, 14, 16 and 18. No doubt we would have obtained many more nonisomorphic graphs with these parameters if we would have continued the search for other graphs for which the 2-rank has increased after switching. But the isomorphism tests are very time consuming, and since we are mainly interested in the 2-ranks, we chose not to do so. We did, however, continue with some other graphs without worrying about isomorphism in the hope to find examples with a 2-rank of 20 (or more), without success.

We will not display all newly obtained strongly regular graphs, instead we just give the sequence of switching sets that increases the 2-rank in each step (vertices are represented as row vectors):

$$\begin{aligned} &\{(100000), (010000), (101000), (011000)\}, \\ &\{(100000), (010000), (100100), (010100)\}, \\ &\{(100000), (010000), (100010), (010010)\}, \\ &\{(100000), (010000), (100001), (010001)\}, \\ &\{(110000), (001000), (000010), (111010)\}, \\ &\{(110000), (001000), (000001), (111001)\}. \end{aligned}$$

The number of cospectral graphs obtained in each of the six above iterations is: 4275 with 2-rank 8 (161 are nonisomorphic), 2238 with 2-rank 10 (195 are nonisomorphic), 1242 with 2-rank 12 (301 are nonisomorphic), 818 with 2-rank 14 (489 are nonisomorphic), 508 with 2-rank 16 (508 are nonisomorphic) and 172 with 2-rank 18 (172 are nonisomorphic).

Thus we see that there is still a gap between the constructed cases and the theoretic upper bound for the 2-rank mentioned in Section 5.3, which for  $Sp(6, 2)$  is 26.

## 5.5 Hadamard matrices and 2-ranks

We recall some results of Hadamard matrices. For more details on Hadamard matrices, see Chapter 18 of [66]. A square  $(+1, -1)$ -matrix  $H$  of order  $n$  is a *Hadamard matrix* (or *H-matrix*) whenever  $HH^T = nI$ . For example

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix} \quad (5.1)$$

is a Hadamard matrix of order 4 (we write  $-$  instead of  $-1$ ). If a row or a column of a Hadamard matrix is multiplied by  $-1$ , it remains a Hadamard matrix. We can multiply rows and columns of any Hadamard matrix by  $-1$  such that the first row and column consist of all ones. Such a Hadamard matrix is called *normalized*. A Hadamard matrix  $H$  is said to be *graphical* if  $H$  is symmetric and it has constant diagonal. Note that if  $H$  is a graphical Hadamard matrix of order  $n$  with  $\delta$  on the diagonal, then  $A = \frac{1}{2}(J - \delta H)$  is the adjacency matrix of a graph on  $n$  vertices. If  $H$  is normalized, the obtained graph has an isolated vertex, and it is well-known that for  $n > 4$  the graph on the remaining  $n - 1$  vertices is strongly regular with parameters  $(n - 1, n/2, n/4, n/4)$ . And conversely, any strongly regular graph with the above parameters comes from a graphical Hadamard matrix. For example, the Hadamard matrix  $H$  in Equation 5.1 is graphical and normalized. The corresponding graph is the smallest symplectic graph  $Sp(2, 2) = K_3$  extended with an isolated vertex. It is well known that if  $H_1$  and  $H_2$  are Hadamard matrices, then so is the Kronecker product  $H_1 \otimes H_2$ . Moreover, if  $H_1$  and  $H_2$  are normalized, then so is  $H_1 \otimes H_2$ , and if  $H_1$  and  $H_2$  are graphical, then so is  $H_1 \otimes H_2$ . For a Hadamard matrix  $H$ , we define  $A_H = \frac{1}{2}(J - H)$  and  $\rho(H) = 2\text{-rank}(A_H)$ .

**Lemma 24.** *Let  $H_1$  and  $H_2$  be two Hadamard matrices, then  $\rho(H_1 \otimes H_2) \leq \rho(H_1) + \rho(H_2)$ , with equality if  $H_1$  and  $H_2$  are normalized.*

*Proof.* It is easily seen that

$$A_{H_1 \otimes H_2} = (J \otimes A_{H_1}) + (A_{H_2} \otimes J) \pmod{2}.$$

For any integer matrix  $A$  we have  $2\text{-rank}(J \otimes A) = 2\text{-rank}(A \otimes J) = 2\text{-rank}(A)$ . Therefore  $\rho(H_1 \otimes H_2) \leq \rho(H_1) + \rho(H_2)$ .

To prove the second statement, we define  $V_i$  to be a matrix consisting of  $\rho(H_i)$  independent columns of  $A_{H_i}$  for  $i = 1, 2$  (so the columns of  $V_1$  and  $V_2$  form a basis for the column space of  $A_{H_1}$  and  $A_{H_2}$ , respectively). Suppose  $H_1$  and  $H_2$  are normalized. Then  $A_{H_1 \otimes H_2}$  contains the columns of  $\mathbf{1} \otimes V_1$  and  $V_2 \otimes \mathbf{1}$ . These  $\rho(H_1) + \rho(H_2)$  columns are independent (indeed, the first rows of  $V_1$  and  $V_2$  are all-zero rows and therefore the only vector in the intersection of the column space of  $\mathbf{1} \otimes V_1$  and the column space of  $V_2 \otimes \mathbf{1}$  is the zero vector), and hence  $\rho(H_1 \otimes H_2) = \rho(H_1) + \rho(H_2)$ .  $\square$

With the Hadamard matrix  $H$  of order 4, given above we define

$$H^{\otimes \nu} = H \otimes H \otimes \cdots \otimes H \text{ } (\nu \text{ times}).$$

Then clearly  $H^{\otimes \nu}$  is a normalized graphical Hadamard matrix of order  $4^\nu$ , and Lemma 24 implies that  $2\text{-rank}(A_{H^{\otimes \nu}}) = \rho(H^{\otimes \nu}) = 2\nu$ . Therefore, by Theorem 20 the strongly regular graph associated with  $H^{\otimes \nu}$  is the symplectic graph  $Sp(2\nu, 2)$ .

In the definition of  $H^{\otimes \nu}$  we can replace any triple product  $H \otimes H \otimes H$  by any other regular graphical Hadamard matrix of order 64. By choosing Hadamard matrices coming from the strongly regular graphs with various 2-ranks found by computer in Section 5.4, we obtain normalized graphical Hadamard matrices of order  $4^\nu$ , and the 2-rank of the associated strongly regular can take all even values between  $2\nu$  and  $2\nu + 12\lfloor \nu/3 \rfloor$ . Thus we find:

**Theorem 25.** *For any even  $r \in [2\nu, 2\nu + 12\lfloor \nu/3 \rfloor]$  there exists a strongly regular graph with parameters  $(2^{2\nu} - 1, 2^{\nu-1}, 2^{\nu-2}, 2^{\nu-2})$  and 2-rank  $r$ .*

Another application of Lemma 24 is the following. There exist strongly regular graphs with parameters  $(35, 18, 9, 9)$  for the 2-ranks 6, 8, 10, 12 and 14; see [39]. Let  $H^*$  be the associated normalized graphical Hadamard matrix of order 36, and let  $H$  be as before, then  $H \otimes H^*$  is associated with a strongly regular graph with parameters  $(143, 72, 36, 36)$ . By Lemma 24 we find that such strongly regular graphs exist for every even 2-rank between 8 and 16.

## 5.6 Remarks

A different construction of graphs with the same parameters as  $Sp(2\nu, 2)$  was given by Munemasa and Vanhove [51]. It would be interesting to know

the 2-rank of their construction. It is claimed in [51] that the construction admits a cyclic difference set, and using Corollary 3.7 from [8], it follows that the 2-rank is a multiple of  $2\nu$ , and therefore at least  $4\nu$ . So we can conclude that their graphs are not isomorphic to the ones obtained in Theorem 23.

A graph associated with a normalized graphical Hadamard matrix, is a so-called  $(v, k, \lambda)$  graph, which means that the adjacency matrix can be interpreted as the incidence matrix of a symmetric  $2$ -( $v, k, \lambda$ ) design. It is possible that nonisomorphic graphs lead to isomorphic designs. However, if the matrices have different 2-ranks, then obviously also the designs are nonisomorphic. Thus we can conclude by Theorem 25 that there exist at least  $1 + 6\lfloor \nu/3 \rfloor$  nonisomorphic symmetric  $2$ -( $2^{2\nu} - 1, 2^{2\nu-1}, 2^{2\nu-2}$ ) designs with distinguishing 2-ranks.

# Some Spectral and Quasi-Spectral Characterizations of Distance-Regular Graphs

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As mentioned earlier, a central issue in spectral graph theory is to study whether or not the spectrum of a graph determines it uniquely or, at least, some of its basic characteristics, see the surveys of Van Dam and Haemers [60, 61]. In particular, much attention has been paid to give spectral or quasi-spectral characterizations of distance-regularity.

A distance-regular graph with diameter  $d$  has  $d + 1$  distinct eigenvalues and its spectrum can be obtained from the intersection array and vice versa. However, in general the spectrum of a graph does not tell us whether it is distance-regular or not. In this paper we will prove new results about when distance-regularity of a graph is a property that can be determined by the spectrum. Other contributions in this area are due to Laskar [46], Cvetković [20], Brouwer and Haemers [13], Van Dam and Haemers [59], Van Dam, Haemers, Koolen, and Spence [63], Haemers [38], and Huang and Liu [44], among others. A survey of the most relevant results can also be found in the textbook of Brouwer and Haemers [14], and the survey of Van Dam, Koolen, and Tanaka [65].

In [59] Van Dam and Haemers gave conditions for distance-regularity under the assumption that the graph is cospectral with a distance-regular graph.

Our aim is to make these conditions less restrictive by using the so-called preintersection numbers, and dropping the assumption that the graph is cospectral with a distance-regular graph. The preintersection numbers are numbers that follow from the spectrum and resemble the intersection numbers of distance-regular graphs. Indeed, we will give new spectral and quasi-spectral characterizations of distance-regularity without requiring, as it is common in this area of research, that:

- $G$  is cospectral with a (feasible) distance-regular graph  $\Gamma$ , and
- $\Gamma$  has intersection numbers, or other combinatorial parameters that satisfy certain properties.

For an overview of such results, see Theorem 29. Instead, we shall show that for some of these results, the same conclusions can be obtained within a much more general setting, i.e., that:

- $G$  has preintersection numbers (and for some results also the average of some intersection numbers) that satisfy certain properties.

For example, Van Dam and Haemers [59] showed that a graph  $G$  is distance-regular if it is cospectral with a distance-regular graph  $\Gamma$  with diameter  $d$  and intersection numbers  $c_1 = \dots = c_{d-1} = 1$ . We generalize this in Theorem 40, by showing that if a graph  $G$  has  $d+1$  distinct eigenvalues and preintersection numbers  $\gamma_1 = \dots = \gamma_{d-1} = 1$ , then  $G$  is distance-regular.

This work was motivated by earlier work in this area, in particular by the odd-girth theorem [62]. This result states that a graph with  $d+1$  distinct eigenvalues and odd-girth  $2d+1$  is distance-regular. We recall that the *odd-girth* of a graph is the length of the shortest odd cycle in the graph, and that the odd-girth follows from the spectrum of the graph. The odd-girth theorem generalizes a result of Huang and Liu [44], who showed that every graph that is cospectral to a generalized Odd graph is distance-regular. In order to obtain our results we will, among others, make use of some results on so-called almost distance-regular graphs by Dalfó, Van Dam, Fiol, Garriga, and Gorissen [24]. An important ingredient of our work is an inequality that is inspired by the spectral excess theorem. This result by Fiol and Garriga [29] (for short proofs, see [57, 28]) states that if for every vertex  $u$ , the number of vertices at distance  $d$  from  $u$  is the same as the so-called spectral excess (which can be expressed in terms of the spectrum), then the graph is distance-regular.

This paper is organized as follows. In Section 6.1, we give some basic background information. Also, we recall by Theorem 29 a result that surveys when a graph that is cospectral with a distance-regular must be distance-regular itself. In Section 6.2 we present a few lemmas about properties of the preintersection numbers that are relevant for the proofs of our main results, which are derived in Section 6.3. In particular, in Section 6.3.1, an alternative formulation of the odd-girth theorem is presented; in Section 6.3.2, we prove distance-regularity for graphs with large girth; in Section 6.3.3, conditions on the preintersection numbers are used to prove distance-regularity. Finally, in Section 6.3.4, we apply the results from Section 6.3.3 to refine the results in Section 6.3.2 for graphs with large girth.

## 6.1 Background

In this section we recall some basic concepts, notation, and results on which our study is based. For more background on spectra of graphs, distance-regular graphs, and their characterizations, see [11, 12, 14, 21, 65, 26, 32]. Throughout this paper,  $G = (V, E)$  denotes a finite, simple, and connected graph with vertex set  $V$ , order  $n = |V|$ , size  $e = |E|$ , and diameter  $D$ . The set (‘sphere’) of vertices at distance  $i = 0, \dots, D$  from a given vertex  $u \in V$  is denoted by  $S_i(u)$ , and we let  $k_i(u) = |S_i(u)|$ . When the numbers  $k_i(u)$  do not depend on the vertex  $u \in V$ , which is the case when the graph is distance-regular, we simply write  $k_i$ . For a regular graph, we sometimes abbreviate the valency  $k_1$  by  $k$ . Recall also that, for every  $i = 0, \dots, D$ , the distance matrix  $A_i$  has entries  $(A_i)_{uv} = 1$  if the distance between  $u$  and  $v$ , denoted  $\text{dist}(u, v)$ , is given by  $\text{dist}(u, v) = i$ , and  $(A_i)_{uv} = 0$  otherwise. Thus,  $A_i$  is the adjacency matrix of the distance- $i$  graph  $G_i$ . In particular,  $A_0 = I$  is the identity matrix,  $A_1 = A$  is the adjacency matrix of  $G$ . Note that  $A_0 + \dots + A_D = J$ , the all-1 matrix.

The spectrum of  $G$  is defined as the spectrum of  $A$ , i.e.,

$$\text{sp } G := \{\lambda_0^{m_0}, \dots, \lambda_d^{m_d}\},$$

where the distinct eigenvalues of  $A$  are ordered decreasingly:  $\lambda_0 > \dots > \lambda_d$ , and the superscripts stand for their multiplicities  $m_i = m(\lambda_i)$ . Note that, since  $G$  is connected,  $m_0 = 1$ , and if  $G$  is regular then  $\lambda_0 = k$ . Throughout the paper,  $d$  will denote the number of distinct eigenvalues minus one.

Let  $\mu$  be the minimal polynomial of  $A$ , that is,  $\mu = \prod_{i=0}^d (x - \lambda_i)$ . Then the *Hoffman polynomial*  $H = n\mu(x)/\mu(\lambda_0)$  characterizes regularity of  $G$  by the



condition  $H(A) = J$  (see Hoffman [43]).

### 6.1.1 Orthogonal polynomials and preintersection numbers

Orthogonal polynomials have been useful in the study of distance-regular graphs. Given a graph  $G$  with adjacency matrix  $A$ , and spectrum  $\{\lambda_0^{m_0}, \dots, \lambda_d^{m_d}\}$ , we consider the scalar product

$$\langle p, q \rangle_G := \frac{1}{n} \operatorname{tr}(p(A)q(A)) = \frac{1}{n} \sum_{i=0}^d m_i p(\lambda_i) q(\lambda_i), \quad p, q \in \mathbb{R}_d[x], \quad (6.1)$$

where the second equation follows from standard properties of the trace. Within the vector space of real symmetric  $n \times n$  matrices, we also use the common scalar product  $\langle M, N \rangle = \frac{1}{n} \operatorname{sum}(M \circ N) = \frac{1}{n} \sum_{i,j=1}^n (M)_{ij} (N)_{ij}$ . Note that  $\langle p, q \rangle_G = \langle p(A), q(A) \rangle$ .

Fiol and Garriga [29] introduced the *predistance polynomials*  $p_0, p_1, \dots, p_d$  as the unique sequence of orthogonal polynomials on  $\mathbb{R}_d[x]$  (so with  $\deg p_i = i$  for  $i = 0, \dots, d$ ) with respect to the scalar product (6.1) that are normalized in such a way that  $\|p_i\|_G^2 = p_i(\lambda_0)$ . Like every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1}, \quad i = 0, \dots, d, \quad (6.2)$$

for certain coefficients  $\alpha_i, \beta_i$ , and  $\gamma_i$ , where  $\beta_{-1} = \gamma_{d+1} = 0$ , and  $p_{-1}$  and  $p_{d+1}$  are undetermined. For convenience, we also define the coefficients  $\gamma_0 = 0$  and  $\beta_d = 0$ .

Some properties of the predistance polynomials and the coefficients  $\alpha_i, \beta_i, \gamma_i$ , for  $i = 0, \dots, d$ , are included in the following result (see Cámara, Fàbrega, Fiol, and Garriga [17]).

**Lemma 26.** *Let  $G$  be a graph with average degree  $\bar{k} = 2e/n$ . Then*

- (i)  $p_0 = 1, p_1 = (\lambda_0/\bar{k})x$ ,
- (ii)  $\alpha_i + \beta_i + \gamma_i = \lambda_0$ , for  $i = 0, \dots, d$ ,
- (iii)  $p_{i-1}(\lambda_0)\beta_{i-1} = p_i(\lambda_0)\gamma_i$ , for  $i = 1, \dots, d$ ,
- (iv)  $p_0 + p_1 + \dots + p_d = H$ , the Hoffman polynomial,

(v) The tridiagonal  $(d+1) \times (d+1)$  ‘recurrence matrix’  $R$  given by

$$R = \begin{bmatrix} \alpha_0 & \gamma_1 & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \gamma_d \\ & & & \beta_{d-1} & \alpha_d \end{bmatrix}$$

has eigenvalues  $\lambda_0, \dots, \lambda_d$ .

We also consider the *preintersection numbers*  $\xi_{ij}^h$ , which are the Fourier coefficients of  $p_i p_j$  in terms of the basis  $\{p_h\}_{0 \leq h \leq d}$ , that is,

$$\xi_{ij}^h = \frac{\langle p_i p_j, p_h \rangle_G}{\|p_h\|_G^2} = \frac{1}{np_h(\lambda_0)} \sum_{r=0}^d m(\lambda_r) p_i(\lambda_r) p_j(\lambda_r) p_h(\lambda_r). \quad (6.3)$$

Note that, in particular, the coefficients of the three-term recurrence (6.2) are  $\alpha_i = \xi_{1,i}^i$ ,  $\beta_i = \xi_{1,i+1}^i$ , and  $\gamma_i = \xi_{1,i-1}^i$ . When  $G$  is distance-regular, the predistance polynomials become the distance polynomials, so that  $p_i(A) = A_i$  and  $p_i(\lambda_0) = k_i$  for  $i = 0, \dots, D$ ; and the preintersection numbers become the intersection numbers  $p_{ij}^h = |S_i(u) \cap S_j(v)|$ , where  $u$  and  $v$  are such that  $\text{dist}(u, v) = h$ . For an arbitrary graph and  $i, j, h \leq D$ , we say that the intersection number  $p_{ij}^h$  is *well-defined* if the numbers  $p_{ij}^h(u, v) = |S_i(u) \cap S_j(v)|$  are the same for all vertices  $u, v$  at distance  $h$ , and, in particular, we write  $a_i = p_{1,i}^i$ ,  $b_i = p_{1,i+1}^i$ , and  $c_i = p_{1,i-1}^i$ . We also consider the average

$$\bar{p}_{ij}^h = \frac{\langle A_i A_j, A_h \rangle}{\|A_h\|^2}$$

of the numbers  $p_{ij}^h(u, v)$  over all (ordered) pairs of vertices  $u, v$  at distance  $\text{dist}(u, v) = h$ , or its particular cases  $\bar{a}_i$ ,  $\bar{b}_i$ , and  $\bar{c}_i$ . Note that  $\bar{k}_i := \frac{1}{n} \sum_{u \in V} k_i(u) = \bar{p}_{ii}^0$ .

### 6.1.2 Partially distance-regular graphs

A graph  $G$  with diameter  $D$  is called *m-partially distance-regular*, for some  $m = 0, \dots, D$ , if its predistance polynomials satisfy  $p_i(A) = A_i$  for every  $i \leq m$  (see Dalfó, Van Dam, Fiol, Garriga, and Gorissen [24]). In particular, every *m-partially distance-regular* with  $m \geq 1$  must be regular. This is

because  $p_1 = (\lambda_0/\bar{k})x$  and, hence,  $p_1(A) = A$  implies  $\bar{k} = \lambda_0$ , a condition that is equivalent to  $G$  being regular (see e.g. Brouwer and Haemers [14]). As an alternative characterization, we have that  $G$  is  $m$ -partially distance-regular when the intersection numbers  $c_i, a_i, b_i$  up to  $c_m$  are well-defined, that is, the distance matrices satisfy the recurrence

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}, \quad i = 0, \dots, m-1.$$

In this case, these intersection numbers are equal to the corresponding preintersection numbers  $\gamma_i, \alpha_i, \beta_i$  up to  $\gamma_m$ . The two following results were derived in [24] by using both characterizations.

**Lemma 27.** *Let  $G$  be a regular graph with girth  $g$ . Then  $G$  is  $m$ -partially distance-regular with  $m = \lfloor (g-1)/2 \rfloor$ .*

**Proposition 28.** *Let  $G$  be a graph with  $d+1$  distinct eigenvalues.*

- (i) *If  $G$  is  $(d-1)$ -partially distance-regular, then  $G$  is distance-regular,*
- (ii) *If  $G$  is bipartite and  $(d-2)$ -partially distance-regular, then  $G$  is distance-regular.*

### 6.1.3 Distance-regularity from cospectrality with a distance-regular graph

It is well-known that every regular graph  $G$  with  $d+1 = 3$  distinct eigenvalues is distance-regular (so, in this case, it is strongly regular). However, when  $d+1 \geq 4$ , only in some special cases it follows from the spectrum of  $G$  that it is distance-regular. The following theorem, given in the recent survey by Van Dam, Koolen, and Tanaka [65] (see also Van Dam and Haemers [60] and Brouwer and Haemers [14]), shows these cases. Note that one of the assumptions is that the graph is cospectral with a distance-regular graph.

**Theorem 29.** *If  $\Gamma$  is a distance-regular graph with diameter  $D = d$  and girth  $g$  satisfying one of the properties (i)-(ix), then every graph  $G$  cospectral with  $\Gamma$  is also distance-regular and  $G$  has the same intersection numbers as  $\Gamma$ .*

- (i)  $g \geq 2d-1$  (Brouwer and Haemers [13]),
- (ii)  $g \geq 2d-2$  and  $\Gamma$  is bipartite (Van Dam and Haemers [59]),
- (iii)  $g \geq 2d-2$  and  $c_{d-1}c_d < -(c_{d-1}+1)(\lambda_1 + \dots + \lambda_d)$  (Van Dam and Haemers [59]),

- (iv)  $\Gamma$  is a generalized Odd graph, i.e.,  $a_1 = \cdots = a_{d-1} = 0$ ,  $a_d \neq 0$  (Huang and Liu [44]),
- (v)  $c_1 = \cdots = c_{d-1} = 1$  (Van Dam and Haemers [59]),
- (vi)  $\Gamma$  is the dodecahedron, or the icosahedron graph (Brouwer and Haemers [13]),
- (vii)  $\Gamma$  is the coset graph of the extended ternary Golay code (Van Dam and Haemers [59]),
- (viii)  $\Gamma$  is the Ivanov-Ivanov-Faradjev graph (Van Dam, Haemers, Koolen, and Spence [63]),
- (ix)  $\Gamma$  is the Hamming graph  $H(3, q)$ , with  $q \geq 36$  (Bang, Van Dam, and Koolen [9]).

## 6.2 Some properties of the preintersection numbers

The main purpose of this section is to derive some properties of the predistance polynomials and preintersection numbers. We will make use of them in order to prove our main results in Section 6.3.

**Lemma 30.** *For  $i = 0, \dots, d$ , the two highest terms of the predistance polynomial  $p_i$  are as in the following expression:*

$$p_i(x) = \frac{1}{\gamma_1 \cdots \gamma_i} [x^i - (\alpha_1 + \cdots + \alpha_{i-1})x^{i-1} + \cdots].$$

*Proof.* Use induction by using the three-term recurrence (6.2) and initial value  $p_0 = 1$ .  $\square$

Note that if the graph is regular, then  $\gamma_1 = 1$  and  $p_1 = x$ .

It is known that the intersection numbers  $a_i$ ,  $b_i$ , and  $c_i$  are nonnegative integers satisfying properties with precise combinatorial meanings (see, for instance, [11, 12]). In contrast, this does not hold for the corresponding preintersection numbers  $\alpha_i$ ,  $\beta_i$ , and  $\gamma_i$ , which in general are not necessarily integral. Nevertheless, the latter do share some of the properties of the former, as shown in Lemma 26 and in the following result.

**Lemma 31.** *Let  $G$  be a graph with distinct eigenvalues  $\lambda_0 > \cdots > \lambda_d$ , and preintersection numbers  $\alpha_i$ ,  $\beta_i$ , and  $\gamma_i$ . Then*

(i)  $\gamma_i > 0$  for  $i = 1, \dots, d$ , and  $\beta_i > 0$  for  $i = 0, \dots, d-1$ ,

(ii)  $\sum_{i=0}^d \alpha_i = \sum_{i=0}^d \lambda_i$ .

*Proof.* (i) First note that  $p_i(\lambda_0) = \|p_i\|_G^2 > 0$  for every  $i = 0, \dots, d$ . Thus, by Lemma 26(iii), we only need to prove the condition on the  $\gamma_i$ 's. Moreover, by the interlacing property of orthogonal polynomials, we know that all the zeros of  $p_i$  lie between  $\lambda_d$  and  $\lambda_0$ . Consequently, the leading coefficient  $\omega_i$  of  $p_i$  must be positive, as  $\lim_{x \rightarrow \infty} p_i(x) = \infty$ . Thus, the conclusion is obtained since by Lemma 30, we have  $\omega_i = (\gamma_1 \cdots \gamma_i)^{-1}$  for  $i = 1, \dots, d$ . To prove (ii) just use Lemma 26(v) and consider the trace of the recurrence matrix  $R$ .  $\square$

In contrast with the above, we know that there are graphs such that  $\lambda_0 + \cdots + \lambda_d < 0$  and, hence, by Lemma 31(ii), some of their preintersection numbers  $\alpha_i$  must be negative. An example is the cubic graph  $G$  with 12 vertices and  $d = 10$  of Figure 6.1 (no. 3.83 in [21]), which has spectrum  $\text{sp } G = \{3^1, 1.7321^1, 1.4812^1, 1.2143^1, 1^2, -0.3111^1, -1^1, -1.5392^1, -1.7321^1, -2.1701^1, -2.6751^1\}$  ( $\lambda_0 + \cdots + \lambda_d = -1$ ) and preintersection numbers as shown in the following table.

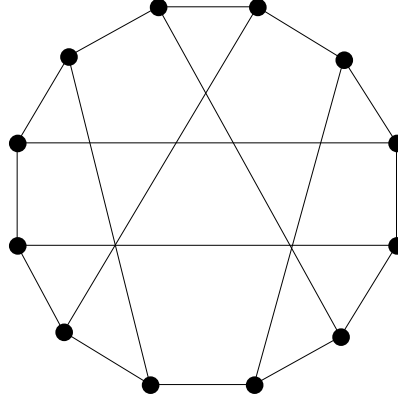
$i$	0	1	2	3	4	5	6	7	8	9	10
$\beta_i$	3	2	1.139	0.434	0.587	0.316	0.254	0.559	0.051	0.671	
$\alpha_i$	0	0	0.750	-0.257	-0.382	0.052	-0.849	-0.097	0.083	-0.570	0.270
$\gamma_i$		1	1.111	2.823	2.795	2.632	3.595	2.538	2.866	2.899	2.730

Note that the  $\alpha_i$ 's sum up to  $-1$ , in accordance with Lemma 31(ii). (Here we should warn the reader that most of the eigenvalues and the entries of the table are not exact but rounded.) Note also that, contrarily to the case of the intersection numbers  $b_i$ 's and  $c_i$ 's, the  $\beta_i$ 's and the  $\gamma_i$ 's do not show a monotone behavior and, even more,  $\gamma_6 > \lambda_0$ .

On the other hand, the given graph does not have triangles and  $\alpha_1 = 0$ . This is not a coincidence: It follows from an inductive argument first used by Van Dam and Haemers [62] that the odd-girth (that is, the length of the shortest odd cycle) can be determined from the preintersection numbers as follows.

**Lemma 32.** *A non-bipartite graph has odd-girth  $2m+1$  if and only if  $\alpha_0 = \cdots = \alpha_{m-1} = 0$  and  $\alpha_m \neq 0$ . A graph is bipartite if and only if  $\alpha_0 = \cdots = \alpha_d = 0$ .*

*Proof.* Let us first assume that  $G$  has odd girth  $2m+1$ . Then  $\text{tr } A^{2i+1} = 0$  for  $i = 0, \dots, m-1$  and  $\text{tr } A^{2m+1} \neq 0$ . Using this, it can be shown by



**Figure 6.1** A cubic graph with negative preintersection numbers.

induction (like in [62]) that  $\alpha_i = 0$  for  $i < m$  and that the predistance polynomials  $p_i$  are odd or even functions depending on whether  $i$  is odd or even, respectively, for  $i \leq m$ . Moreover,

$$\alpha_m = \frac{1}{p_m(\lambda_0)} \langle xp_m, p_m \rangle_G = \frac{1}{np_m(\lambda_0)} \text{tr}(Ap_m^2(A)) \neq 0,$$

since the polynomial  $xp_m^2$  is an odd function and has degree  $2m + 1$ , so the leading term is the only one contributing to the trace.

Conversely, assume that the preintersection numbers satisfy  $\alpha_i = 0$  for  $i = 0, \dots, m - 1$  and  $\alpha_m \neq 0$ . Then again, by (6.2), the parity of the predistance polynomial  $p_i$  (that is, it is an odd or even function) coincides with the parity of its index  $i$  for  $i = 0, \dots, m$ . Then, for any  $i < m$  we have that  $\text{tr} A^{2i+1} = n \langle A^i, A^{i+1} \rangle = n \langle x^i, x^{i+1} \rangle_G = 0$ , as the expressions of  $x^i$  and  $x^{i+1}$  in terms of the basis  $p_0, \dots, p_m$  have polynomials with distinct parity. Thus,  $G$  has no odd cycles of length smaller than  $2m + 1$ , and since  $\alpha_m \neq 0$ , it follows (from the first part of the proof) that the odd-girth is indeed  $2m + 1$ .

The statement about bipartiteness follows from using parts of the above arguments.  $\square$

Note that in general, the girth is not determined by the spectrum, but for regular graphs it is. In Corollary 42 we will make this explicit in terms of the preintersection numbers.

### 6.3 New quasi-spectral characterizations of distance-regular graphs

This section contains the main results of our work. As mentioned in the introduction, we give sufficient conditions for distance-regularity of a graph  $G$ , without requiring  $G$  to be cospectral with a distance-regular graph. We begin with an alternative formulation of the so-called odd-girth theorem [62].

#### 6.3.1 The odd-girth theorem revisited

Theorem 29(iv) was generalized by Van Dam and Haemers [62] as the odd-girth theorem, which states that a graph  $G$  with  $d + 1$  distinct eigenvalues and odd-girth  $2d + 1$  is distance-regular. By our Lemma 32, the condition on the odd-girth of  $G$  is equivalent to  $\alpha_1 = \cdots = \alpha_{d-1} = 0$ ,  $\alpha_d \neq 0$ , which corresponds to the condition  $a_1 = \cdots = a_{d-1} = 0$ ,  $a_d \neq 0$  of Theorem 29(iv). Note that Lee and Weng [47], and Van Dam and Fiol [58] showed that the odd-girth theorem is not restricted to regular graphs.

Before presenting an alternative formulation of the odd-girth theorem, recall that a *generalized Odd graph* is a distance-regular graph with diameter  $D$  and odd-girth  $2D + 1$ . A well-know example is the *Odd graph*  $O_k$ , whose vertices represent the  $(k - 1)$ -element subsets of a  $(2k - 1)$ -element set, where two vertices are adjacent if and only if their corresponding subsets are disjoint, see Biggs [11].

**Theorem 33.** *Let  $G$  be a non-bipartite graph with  $d + 1$  distinct eigenvalues.*

*(i) If  $\alpha_i \geq 0$  for  $i = 0, \dots, d - 1$ , then*

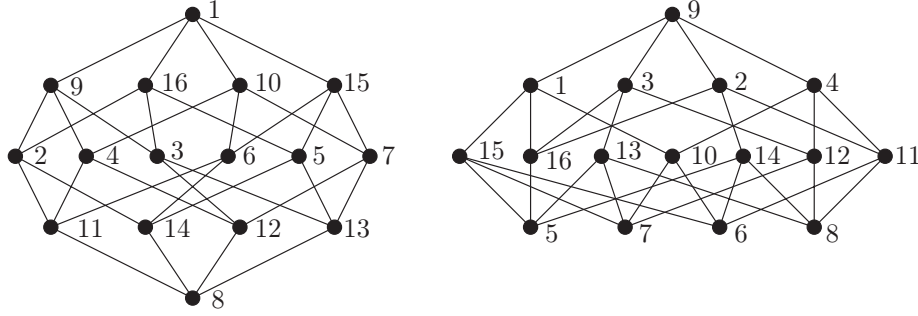
$$\gamma_d \geq -(\lambda_1 + \cdots + \lambda_d),$$

*with equality if and only if  $G$  is a distance-regular generalized Odd graph.*

*(ii) If  $G$  has odd-girth at least  $2d - 1$  and  $\gamma_d = -(\lambda_1 + \cdots + \lambda_d)$ , then  $G$  is a distance-regular generalized Odd graph.*

*Proof.* We will use that  $\alpha_0 + \cdots + \alpha_d = \lambda_0 + \cdots + \lambda_d$  (by Lemma 31(ii)) and  $\alpha_d + \gamma_d = \lambda_0$  (by Lemma 26(ii) and recalling that  $\beta_d = 0$ ). To show (i), observe that the hypothesis now implies that

$$\gamma_d = \lambda_0 - \alpha_d = -(\lambda_1 + \cdots + \lambda_d) + (\alpha_0 + \cdots + \alpha_{d-1}) \geq -(\lambda_1 + \cdots + \lambda_d),$$



**Figure 6.2** Two views of the Hoffman graph.

with equality if and only if  $\alpha_0 = \dots = \alpha_{d-1} = 0$ . Because  $G$  is not bipartite, this is equivalent to the odd-girth of  $G$  being  $2d + 1$ , and so (i) follows from the odd-girth theorem.

To show (ii), note that by Lemma 32 we have that  $\alpha_0 = \dots = \alpha_{d-2} = 0$ , and hence  $\alpha_{d-1} + \alpha_d = \lambda_0 + \dots + \lambda_d$ . This implies that

$$\gamma_d - \alpha_{d-1} = -(\lambda_1 + \dots + \lambda_d), \quad (6.4)$$

and so, by the assumption,  $\alpha_{d-1} = 0$ . Hence  $G$  has odd-girth  $2d + 1$ , and (ii) follows, again by the odd-girth theorem.  $\square$

We will make further use of (6.4) in the later sections on graphs with large girth. There (Theorem 43) we will also present a variation of Theorem 33(i).

Of course, one of the cases (but certainly not the only one) where the hypothesis that  $\alpha_i \geq 0$  for  $i = 0, \dots, d-1$  holds, is when  $G$  is cospectral with a distance-regular graph. We recall however that the hypothesis is not satisfied in general, see the graph of Figure 6.1.

In contrast with the above, if  $G$  is bipartite, then  $\gamma_d = -(\lambda_1 + \dots + \lambda_d)$ , but in general we cannot conclude that  $G$  is distance-regular. For instance, a counterexample is the Hoffman graph [43], shown in Figure 6.2, which is cospectral with the distance-regular 4-cube  $Q_4$  and hence it is bipartite with  $d = 4$  ( $\alpha_0 = \dots = \alpha_4 = 0$ ). The Hoffman graph is not distance-regular however.



### 6.3.2 Distance-regularity from large girth

The first two cases of Theorem 29 can be generalized as follows.

**Theorem 34.** *A regular graph  $G$  with girth  $g$  is distance-regular if any of the following condition holds:*

- (i)  $g \geq 2d - 1$ ,
- (ii)  $g \geq 2d - 2$  and  $G$  is bipartite.

*Proof.* (i) If  $g \geq 2d - 1$ , then  $G$  is  $(d - 1)$ -partially distance-regular by Lemma 27, and the result follows from Proposition 28(i). The proof of (ii) is similar by using Proposition 28(ii).  $\square$

We recall that the condition of being bipartite follows from the spectrum and also the girth of a regular graph is determined by the spectrum, so the assumptions in this result only depend on the spectrum of  $G$ .

As a consequence, and since a bipartite graph has girth  $g \geq 4$ , we obtain the following known results (see, for instance, Abiad, Dalfó, and Fiol [2, 3]).

**Corollary 35.** *Let  $G$  be a regular bipartite graph.*

- (i) *If  $G$  has  $d + 1 = 4$  distinct eigenvalues, then it is distance-regular,*
- (ii) *If  $G$  has  $d + 1 = 5$  distinct eigenvalues and every pair of vertices at distance two has the same number of common neighbors, then it is distance-regular.*

*Proof.* (i) This follows immediately from Theorem 34(ii). The condition on the number of common neighbors in (ii) implies that  $c_2$  is well-defined and hence that  $G$  is 2-partially distance-regular. By Proposition 28(ii), it then follows that  $G$  is distance-regular.  $\square$

### 6.3.3 Distance-regularity from the (pre)intersection numbers

In this section we show how the preintersection numbers can be used to prove distance-regularity. With this aim, we give some properties of the preintersection numbers of  $(m - 1)$ -partially distance-regular graphs. (The case (i) was also proved in [23, Prop. 1(c)].)

**Lemma 36.** *Let  $G$  be a regular graph and let  $m \leq D$  be a positive integer. Suppose that  $G$  is  $(m-1)$ -partially distance-regular. Then*

- (i)  $\alpha_{m-1} = \bar{a}_{m-1}$  and  $\beta_{m-1} = \bar{b}_{m-1} = \frac{\bar{k}_m \bar{c}_m}{\bar{k}_{m-1}}$ ,
- (ii)  $k_{m-1} \alpha_{m-1}^2 + p_m(\lambda_0) \gamma_m^2 = k_{m-1} \bar{a}_{m-1}^2 + \bar{k}_m \bar{c}_m^2$ ,
- (iii)  $p_m(\lambda_0) \gamma_m^2 \geq \bar{k}_m \bar{c}_m^2$ , with equality if and only if  $a_{m-1}$  is well-defined,
- (iv) If  $a_{m-1}$  is well defined, then  $\gamma_m = \frac{\bar{c}_m^2}{\bar{c}_m}$ .

*Proof.* Note first that, since  $G$  is  $(m-1)$ -partially distance-regular, all its intersections numbers up to  $c_{m-1}$  are well-defined. Then the following computation proves the first part of (i):

$$\begin{aligned} \alpha_{m-1} &= \frac{\langle p_1 p_{m-1}, p_{m-1} \rangle_G}{\|p_{m-1}\|_G^2} = \frac{1}{\|A_{m-1}\|^2} \langle AA_{m-1}, A_{m-1} \rangle \\ &= \frac{1}{nk_{m-1}} \sum_{u,v \in V} (AA_{m-1})_{uv} (A_{m-1})_{uv} \\ &= \frac{1}{nk_{m-1}} \sum_{\text{dist}(u,v)=m-1} a_{m-1}(u,v) = \bar{a}_{m-1}. \end{aligned}$$

The equalities  $\beta_{m-1} = k - c_{m-1} - \alpha_{m-1} = \bar{b}_{m-1}$  follow from the above, the regularity assumption, and Lemma 26(ii). Moreover, by counting in two ways the total number of edges between  $S_{m-1}(u)$  and  $S_m(u)$  for all  $u \in V$ , see Figure 6.3(a), we get:

$$nk_{m-1} \bar{b}_{m-1} = \sum_{u \in V} \sum_{v \in S_m(u)} c_m(u,v).$$

Hence,

$$\bar{c}_m = \frac{1}{n\bar{k}_m} \sum_{u \in V} \sum_{v \in S_m(u)} c_m(u,v) = \frac{1}{\bar{k}_m} k_{m-1} \bar{b}_{m-1},$$

whence the second equality for  $\beta_{m-1} = \bar{b}_{m-1}$  in (i) follows. Here we remark that  $\bar{k}_m > 0$  because  $m \leq D$ , but the values of  $c_m(u,v)$  are really only used for the case of a vertex  $v$  at distance  $m$  from a vertex  $u$ . We do not require that every vertex  $u$  has a vertex at distance  $m$ . Note also that  $k_{m-1} = p_{m-1}(\lambda_0)$ .

(ii) From (6.2) with  $i = m - 1$ , and because  $G$  is  $(m - 1)$ -partially distance-regular (which implies that  $p_i(A) = A_i$  for  $i \leq m - 1$ ), it follows that

$$\alpha_{m-1}p_{m-1}(A) + \gamma_m p_m(A) = AA_{m-1} - b_{m-2}A_{m-2}. \quad (6.5)$$

Moreover, by (6.3),

$$\alpha_{m-1} = \xi_{1,m-1}^{m-1} = \frac{\langle xp_{m-1}, p_{m-1} \rangle_G}{\|p_{m-1}\|_G^2} \quad \text{and} \quad \gamma_m = \xi_{1,m-1}^m = \frac{\langle xp_{m-1}, p_m \rangle_G}{\|p_m\|_G^2}$$

with  $\|p_{m-1}\|_G^2 = p_{m-1}(\lambda_0) = k_{m-1}$ , and  $\|p_m\|_G^2 = p_m(\lambda_0)$ . Therefore,

$$\begin{aligned} nk_{m-1}\alpha_{m-1}^2 + np_m(\lambda_0)\gamma_m^2 &= n\alpha_{m-1}\langle xp_{m-1}, p_{m-1} \rangle_G + n\gamma_m\langle xp_{m-1}, p_m \rangle_G \\ &= n\langle xp_{m-1}, \alpha_{m-1}p_{m-1} + \gamma_m p_m \rangle_G \\ &= n\langle AA_{m-1}, AA_{m-1} - b_{m-2}A_{m-2} \rangle \\ &= \text{tr}(AA_{m-1})^2 - b_{m-2}\text{tr}(AA_{m-1}A_{m-2}) \\ &= \sum_{u \in V} \sum_{v \in S_{m-1}(u)} a_{m-1}(u, v)^2 + \sum_{u \in V} \sum_{v \in S_m(u)} c_m(u, v)^2 \\ &= nk_{m-1}\overline{a_{m-1}^2} + n\overline{k_m c_m^2}, \end{aligned}$$

where we used (6.5) for the third equality, whereas for the fifth equality we used that

$$\begin{aligned} \text{tr}(AA_{m-1})^2 &= \text{sum}(AA_{m-1} \circ AA_{m-1}) \\ &= \sum_{u \in V} \sum_{v \in S_{m-2}(u) \cup S_{m-1}(u) \cup S_m(u)} (AA_{m-1})_{uv}^2 \\ &= nk_{m-2}b_{m-2}^2 + \sum_{u \in V} \sum_{v \in S_{m-1}(u)} a_{m-1}(u, v)^2 + \sum_{u \in V} \sum_{v \in S_m(u)} c_m(u, v)^2, \end{aligned}$$

and similarly that  $\text{tr}(AA_{m-1}A_{m-2}) = \text{sum}(AA_{m-1} \circ A_{m-2}) = nk_{m-2}b_{m-2}$ .

(iii) By using (i), it follows that

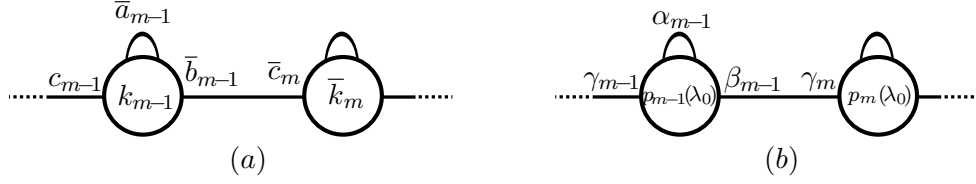
$$\overline{a_{m-1}^2} \geq (\overline{a_{m-1}})^2 = \alpha_{m-1}^2,$$

with equality if and only if  $a_{m-1}$  is well-defined (and  $a_{m-1} = \alpha_{m-1}$ ). The statement now follows from combining this with (ii).

(iv) Using (i), we obtain  $p_m(\lambda_0)\gamma_m = p_{m-1}(\lambda_0)\beta_{m-1} = k_{m-1}\beta_{m-1} = \overline{k_m c_m}$ . Thus, from this and (iii),

$$\overline{k_m c_m^2} = p_m(\lambda_0)\gamma_m^2 = \overline{k_m c_m}\gamma_m,$$

whence the result follows.  $\square$



**Figure 6.3** The (pre)intersection numbers and their averages.

The following observation is the key to many of our results. It is motivated by the spectral excess theorem and we will use it to prove Proposition 38. From there, we will derive several spectral and quasi-spectral characterizations of distance-regularity.

**Proposition 37.** *Let  $G$  be a regular graph and let  $m \leq D$  be a positive integer. If  $G$  is  $(m-1)$ -partially distance-regular, then  $\bar{k}_m \geq p_m(\lambda_0)$  with equality if and only if  $G$  is  $m$ -partially distance-regular.*

*Proof.* By the assumption,  $p_i(A) = A_i$  for  $i < m$ . Moreover,  $(p_m(A))_{uv} = 0$  for every pair of vertices  $u, v$  at distance  $i > m$  and hence  $\langle p_m(A), A_i \rangle = 0$ . This implies that

$$\begin{aligned} \langle p_m(A), A_m \rangle &= \langle p_m(A), \sum_{i < m} p_i(A) + A_m + \sum_{i > m} A_i \rangle \\ &= \langle p_m(A), J \rangle = \langle p_m, H \rangle_G = \langle p_m, p_0 + \cdots + p_d \rangle_G = \|p_m\|_G^2 = p_m(\lambda_0), \end{aligned}$$

where we used Lemma 26(iv). Then, by the Cauchy-Schwarz inequality,  $p_m^2(\lambda_0) \leq \|p_m(A)\|^2 \|A_m\|^2 = p_m(\lambda_0) \bar{k}_m$ , and hence  $\bar{k}_m \geq p_m(\lambda_0)$ . Furthermore, in the case of equality,  $p_m(A) = \alpha A_m$  for some  $\alpha \in \mathbb{R}$ , and by taking norms we get that  $\alpha = 1$  since  $p_m(\lambda_0) > 0$ .  $\square$

Now we are ready to give the following result, which generalizes some fundamental results by Van Dam and Haemers [59], and Van Dam, Haemers, Koolen, and Spence [63].

**Proposition 38.** *Let  $G$  be a regular graph and let  $m \leq D$  be a positive integer. Suppose that  $G$  is  $(m-1)$ -partially distance-regular and any of the following conditions holds:*

$$(i) \quad \bar{c}_m \geq \gamma_m,$$

- (ii)  $c_{m-1} \geq \gamma_m$ ,
- (iii)  $k_{m-1}(\overline{a_{m-1}^2} - \alpha_{m-1}^2) + \bar{k}_m(\overline{c_m^2} - \gamma_m^2) \geq 0$ ,
- (iv)  $\overline{c_m^2} \geq \gamma_m^2$ ,
- (v)  $a_{m-1}$  is well-defined and  $c_m(u, v) \leq \gamma_m$  for every pair of vertices  $u, v$  at distance  $m$ .

Then  $G$  is  $m$ -partially distance-regular with intersection numbers  $a_{m-1} = \alpha_{m-1}$  and  $c_m = \gamma_m$ .

*Proof.* (i) By Lemma 36(i) and the hypothesis,

$$\bar{k}_m = \frac{1}{\bar{c}_m} p_{m-1}(\lambda_0) \beta_{m-1} = \frac{1}{\bar{c}_m} p_m(\lambda_0) \gamma_m \leq p_m(\lambda_0),$$

where we also used Lemma 26(iii), see Figure 6.3(b). Now the conclusion follows from Proposition 37.

(ii) This is a simple consequence of (i) since, for every pair of vertices  $u, v$  at distance  $m$ , it follows that  $c_m(u, v) \geq c_{m-1}(u', v) = c_{m-1} \geq \gamma_m$ , where  $u, u', \dots, v$  is a shortest path.

(iii) By Lemma 36(ii) and the hypothesis, we have

$$p_m(\lambda_0) = \frac{k_{m-1}(\overline{a_{m-1}^2} - \alpha_{m-1}^2) + \bar{k}_m \overline{c_m^2}}{\gamma_m^2} \geq \frac{\bar{k}_m \gamma_m^2}{\gamma_m^2} = \bar{k}_m,$$

and the result follows from Proposition 37.

(iv) From Lemma 36(iii) and the hypothesis, we have that  $p_m(\lambda_0) \geq \bar{k}_m$ , and the result follows again from Proposition 37.

(v) From the hypothesis, we get  $\overline{c_m^2} \leq \gamma_m \bar{c}_m$ , but from Lemma 36(iv), this must be an equality. Therefore, the intersection number  $c_m$  is also well-defined and equal to  $\gamma_m$ , which proves the result.  $\square$

Since  $\overline{c_m^2} \geq (\bar{c}_m)^2$ , the result with condition (i) is a consequence of the result involving condition (iv). Also, observe that, because of Lemma 36(i), the proof of Proposition 38(v) also works if we change the hypothesis ‘ $a_{m-1}$  is well-defined’ to either ‘ $a_{m-1}(u, v) \leq \alpha_{m-1}$  for every  $u, v$  at distance  $m-1$ ’ or ‘ $a_{m-1}(u, v) \geq \alpha_{m-1}$  for every  $u, v$  at distance  $m-1$ ’. The result also holds if we require that ‘ $c_m(u, v) \geq \gamma_m$  for every  $u, v$  at distance  $m$ ’, in which case

we do not need the above hypotheses on  $a_{m-1}$  since then  $\bar{c}_m \geq \gamma_m$  and the result follows from Proposition 38(i).

As a consequence of Proposition 38(i), and since every regular graph is clearly 1-partially distance-regular with  $c_1 = \gamma_1 = 1$ , we have the following result.

**Theorem 39.** (i) *Every regular graph  $G$  with  $D \geq d - 1$  and preintersection numbers satisfying  $\bar{c}_i \geq \gamma_i$  for  $i = 2, \dots, d - 1$ , is distance-regular,*

(ii) *Every regular bipartite graph  $G$  with  $D \geq d - 2$  and preintersection numbers satisfying  $\bar{c}_i \geq \gamma_i$  for  $i = 2, \dots, d - 2$ , is distance-regular.*

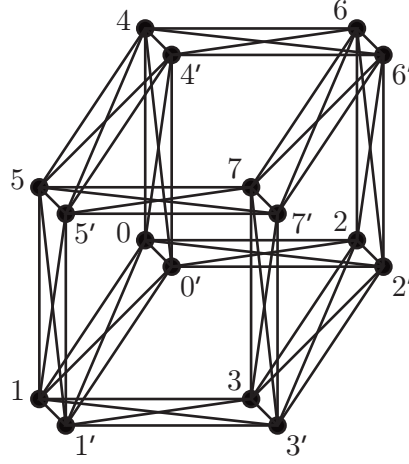
*Proof.* Apply recursively Proposition 38(i) to show that  $G$  is  $(d - 1)$ -partially (respectively  $(d - 2)$ -partially) distance-regular and use Proposition 28(i) (respectively, 28(ii)).  $\square$

From Theorem 39(i), it clearly follows that if  $G$  has the parameters  $c_i$  well-defined and equal to  $\gamma_i$  for  $i = 1, \dots, d - 1$ , then  $G$  is distance-regular. Note that it is not enough to assume only that the  $c_i$ 's are well-defined. To illustrate this, we give an example of a non-distance-regular graph with well-defined  $k_i$  and  $c_i$ . Consider the strong product  $G$  of the cube  $Q_3$  with the complete graph  $K_2$ , shown in Figure 6.4. This graph is 7-regular with spectrum  $\text{sp } G = \{7^1, 3^3, -1^{11}, -5^1\}$ , it has diameter  $D = d = 3$ , and well-defined intersection numbers  $c_1 = 1$ ,  $c_2 = 4$ , and  $c_3 = 6$ . However, it is not a distance-regular graph. (Note that  $G$  has preintersection numbers  $\gamma_1 = 1$ ,  $\gamma_2 = 4.571$  and  $\gamma_3 = 4.816$ .) Even more so, it has well-defined  $k_1 = 7$ ,  $k_2 = 6$ , and  $k_3 = 2$  (which is easily seen because  $G$  is vertex-transitive). In fact, only  $a_1$  and  $b_1$  are not well-defined.

Similarly, if you take the Kronecker product of the adjacency matrix of a bipartite distance-regular graph with even diameter  $D$  with the all-one matrix  $J_2$ , then the result is the adjacency matrix of a regular graph with diameter  $D = d$  and with well-defined  $k_i$  and  $a_i$ , but it is not distance-regular, since  $c_2$  and  $b_2$  are not well-defined.

These examples show that the combinatorics is not sufficient and some extra spectral information is required. This is in line with earlier results in the literature, where cospectrality with a distance-regular graph, or feasible spectrum for a distance-regular graph, is required (see, for example, Haemers [38] or Van Dam and Haemers [59]).

Another consequence of Proposition 38 is the following result. It corresponds to the result of Van Dam and Haemers [59] stated in Theorem 29(v), and



**Figure 6.4** The strong product of  $Q_3$  by  $K_2$ .

its bipartite counterpart. Recall that the preintersection numbers are determined by the spectrum, and that regularity of a graph is characterized by the condition that  $\gamma_1 = 1$ .

**Theorem 40.** *Let  $G$  be a graph with  $d + 1$  distinct eigenvalues.*

- (i) *If  $d \geq 2$  and  $G$  has preintersection numbers  $\gamma_1 = \cdots = \gamma_{d-1} = 1$ , then it is distance-regular,*
- (ii) *If  $d \geq 3$  and  $G$  is bipartite and has preintersection numbers  $\gamma_1 = \cdots = \gamma_{d-2} = 1$ , then it is distance-regular.*

*Proof.* (i) If  $D \leq d-1$ , then apply Proposition 38(i) or (ii) recursively (using that  $\bar{c}_m \geq 1$  and  $c_{m-1} \geq 1$ ) to derive that  $G$  is  $D$ -partially distance-regular, that is, that  $G$  is distance-regular. If  $D = d$ , then it follows similarly that  $G$  is  $(d-1)$ -partially distance-regular, and then it follows from Proposition 28(i) that  $G$  is distance-regular. The proof of (ii) is similar.  $\square$

Moreover, Proposition 38(ii) also yields the following slight improvement of Proposition 28. Recall that 1-partial distance-regularity implies regularity.

**Proposition 41.** *Let  $G$  be a graph with  $d + 1$  distinct eigenvalues.*

- (i) If  $d \geq 3$ ,  $G$  is  $(d-2)$ -partially distance-regular, and  $\gamma_{d-1} \leq c_{d-2}$ , then  $G$  is distance-regular,
- (ii) If  $d \geq 4$ ,  $G$  is bipartite and  $(d-3)$ -partially distance-regular, and  $\gamma_{d-2} \leq c_{d-3}$ , then  $G$  is distance-regular.

To conclude this subsection, we also give a characterization of the girth of a regular graph in terms of the preintersection numbers (cf. Lemma 32 for a similar characterization for the odd-girth).

**Corollary 42.** (i) A regular graph has girth  $2m+1$  if and only if  $\alpha_0 = \dots = \alpha_{m-1} = 0$ ,  $\alpha_m \neq 0$ , and  $\gamma_1 = \dots = \gamma_m = 1$ ,  
(ii) A regular graph has girth  $2m$  if and only if  $\alpha_0 = \dots = \alpha_{m-1} = 0$ ,  $\gamma_1 = \dots = \gamma_{m-1} = 1$ , and  $\gamma_m > 1$ .

*Proof.* This follows from combining Lemma 32 and Proposition 38(ii) recursively.  $\square$

### 6.3.4 Distance-regularity from large girth revisited

Our aim here is to give some improvements of the results in Section 6.3.2 for graphs with large girth. First, from Proposition 38(v), we obtain a refinement of the results in Theorems 29(iii) and 34.

**Theorem 43.** Let  $G$  be a regular graph with  $d+1$  distinct eigenvalues  $\lambda_0 > \dots > \lambda_d$  and girth  $g \geq 2d-2$ . Then

$$\gamma_d \geq -(\lambda_1 + \dots + \lambda_d), \quad (6.6)$$

with equality if and only if  $G$  is distance-regular and either bipartite or a generalized Odd graph.

*Proof.* Note that, from the hypothesis on the girth,  $G$  is  $(d-2)$ -partially distance-regular with  $c_i = \gamma_i = 1$  and  $a_i = \alpha_i = 0$  for  $i = 1, \dots, d-2$  (Lemma 27 and Corollary 42). Moreover, since the Hoffman polynomial is

$$H(x) = \sum_{i=0}^d p_i(x) = \frac{n}{\pi_0} \prod_{i=1}^d (x - \lambda_i) = \frac{n}{\pi_0} [x^d - (\lambda_1 + \dots + \lambda_d)x^{d-1} + \dots],$$

where  $\pi_0 = \prod_{i=1}^d (\lambda_0 - \lambda_i)$ , the leading coefficient of  $p_d$  is  $\omega_d = (\gamma_d \gamma_{d-1})^{-1} = n/\pi_0$  (the first equality comes from the three-term recurrence (6.2)). Now, if



we consider two vertices  $u, v$  at distance  $d-1$ , then the Hoffman polynomial, satisfying  $H(A) = J$ , yields

$$1 = \frac{n}{\pi_0}[(A^d)_{uv} - (\lambda_1 + \cdots + \lambda_d)(A^{d-1})_{uv}].$$

Hence,

$$(\lambda_1 + \cdots + \lambda_d)(A^{d-1})_{uv} + \gamma_{d-1}\gamma_d = (A^d)_{uv} \geq 0. \quad (6.7)$$

Now let us assume, contrary to (6.6), that  $\gamma_d \leq -(\lambda_1 + \cdots + \lambda_d)$ , and aim to prove equality. Then, using the fact that  $\gamma_d > 0$  (Lemma 31(i)), we have

$$c_{d-1}(u, v) = (A^{d-1})_{uv} \leq \frac{\gamma_{d-1}\gamma_d}{-(\lambda_1 + \cdots + \lambda_d)} \leq \gamma_{d-1}. \quad (6.8)$$

Consequently, from Proposition 38(v),  $G$  is  $(d-1)$ -partially distance-regular, and by using Proposition 28(i), we conclude that  $G$  is distance-regular with  $c_{d-1} = \gamma_{d-1}$ . Then, equalities in (6.8) hold for all vertices  $u, v$  at distance  $d-1$ , and we are in the case of equality:  $\gamma_d = -(\lambda_1 + \cdots + \lambda_d)$ . Moreover, this holds if and only if  $(A^d)_{uv} = 0$  in (6.7), which means that there are no odd cycles of length smaller than  $2d+1$ , so  $a_0 = \cdots = a_{d-1} = 0$ , and  $G$  is either bipartite or a generalized Odd graph. Conversely, when  $G$  is bipartite, we have  $\gamma_d = c_d = -\lambda_d = \lambda_0$  (the degree of  $G$ ), and the condition (6.6) is tight. Moreover, when  $G$  is a generalized Odd graph, with odd-girth  $2d+1$ , Van Dam and Haemers [62] proved that  $\alpha_d = a_d = \lambda_0 + \cdots + \lambda_d$  (this is also a consequence of Lemma 31(ii)), and equality in (6.6) follows again from  $\alpha_d + \gamma_d = \lambda_0$  (Lemma 26(ii)).  $\square$

Note that, as a consequence of Theorem 43, the assumptions of Theorem 29(iii) seem to be quite strong.

An alternative reasoning that suggests (6.6) is the following. Since the odd-girth of  $G$  is at least  $2d-1$ , it follows by (6.4) that (6.6) is equivalent to  $\alpha_{d-1} \geq 0$ .

By using Proposition 38(i), we can also obtain some related results. With this aim, let  $\bar{a}_{d-1}^{(d)}$  be the mean number of walks of length  $d$  between vertices at distance  $d-1$ .

**Proposition 44.** *Let  $G$  be a regular graph with  $d+1$  distinct eigenvalues  $\lambda_0 > \cdots > \lambda_d$  and girth  $g \geq 2d-2$ .*

- (i) *If  $\alpha_{d-1} < \gamma_d$ , then  $\bar{a}_{d-1}^{(d)} \geq \alpha_{d-1}\gamma_{d-1}$ , with equality if and only if  $G$  is distance-regular,*

(ii) If  $\alpha_{d-1} > \gamma_d$ , then  $\bar{a}_{d-1}^{(d)} \leq \alpha_{d-1}\gamma_{d-1}$ , with equality if and only if  $G$  is distance-regular,

(iii) If  $\alpha_{d-1} = \gamma_d$ , then  $a_{d-1}^{(d)} = \alpha_{d-1}\gamma_{d-1}$  is well-defined.

*Proof.* Necessity in (i) and (ii) is clear since, when  $G$  is distance-regular,  $\alpha_{d-1} = a_{d-1}$ ,  $\gamma_{d-1} = c_{d-1}$ , and, from the hypothesis on the girth,  $\gamma_i = c_i = 1$  for  $i = 1, \dots, d-2$ . So, the number of  $d$ -walks between every pair of vertices  $u, v$  at distance  $d-1$  is  $a_{d-1}c_{d-1}$ .

On the other hand, (6.7) and (6.4) imply that if  $u, v$  are two vertices at distance  $d-1$ , then

$$(\alpha_{d-1} - \gamma_d)c_{d-1}(u, v) + \gamma_{d-1}\gamma_d = a_{uv}^{(d)}. \quad (6.9)$$

Thus, by taking averages over all vertices  $u, v$  at distance  $d-1$ , we have

$$(\alpha_{d-1} - \gamma_d)\bar{c}_{d-1} + \gamma_{d-1}\gamma_d = \bar{a}_{d-1}^{(d)}.$$

Now, for proving sufficiency in the case (i), let us assume that  $\bar{a}_{d-1}^{(d)} \leq \alpha_{d-1}\gamma_{d-1}$ , and aim to prove equality. Then by the hypothesis that  $\alpha_{d-1} < \gamma_d$ , we obtain that

$$\bar{c}_{d-1} = \frac{\gamma_{d-1}\gamma_d - \bar{a}_{d-1}^{(d)}}{\gamma_d - \alpha_{d-1}} \geq \frac{\gamma_{d-1}\gamma_d - \alpha_{d-1}\gamma_{d-1}}{\gamma_d - \alpha_{d-1}} = \gamma_{d-1}.$$

Then, by Proposition 38(i),  $G$  is  $(d-1)$ -partially distance-regular, and the result follows from Proposition 28(i). The proof of sufficiency for the case (ii) is similar.

Finally, if the hypothesis in (iii) holds, then (6.9) gives

$$a_{uv}^{(d)} = (A^d)_{uv} = \gamma_{d-1}\gamma_d = \gamma_{d-1}\alpha_{d-1}$$

for every pair of vertices  $u, v$  at distance  $d-1$  and, hence,  $a_{d-1}^{(d)} = \alpha_{d-1}\gamma_{d-1}$ , as claimed.  $\square$

Note that in Proposition 44(iii), it remains open whether the graph must be distance-regular or not. In fact, it is not easy to find graphs satisfying the conditions of this case. Such an example is the Perkel graph [53] (see also [12, § 13.3]), which is a distance-regular graph with  $n = 57$  vertices, diameter  $D = 3$ , intersection array  $\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{6, 5, 2; 1, 1, 3\}$ , and spectrum  $\{6^1, ((3 + \sqrt{5})/2)^{18}, ((3 - \sqrt{5})/2)^{18}, -3^{20}\}$ . Note that  $\alpha_2 =$

$\gamma_3 = 3$ , as required in the case (iii) of the above result. Moreover, since  $\alpha_1 = 0$  and  $\gamma_2 = 1$ , it has girth  $g = 5 = 2d - 1$ , so it also satisfies the conditions of Theorem 34(i), and hence any graph with the same spectrum is distance-regular (in fact, it is known that this graph is determined by the spectrum, see [63]).

Another—putative—graph suggest that the graphs in this case need not be distance-regular. It is the first relation in a putative 3-class association scheme on 81 vertices, the parameters of which occur on top of p. 102 in the list of [56] (with the second relation being the Brouwer-Haemers graph). The spectrum is  $\{10^1, 1^{20}, (-\frac{1}{2} + \frac{1}{2}\sqrt{45})^{30}, (-\frac{1}{2} - \frac{1}{2}\sqrt{45})^{30}\}$ , and it follows that the (relevant) preintersection numbers are  $\alpha_1 = 0$  (so  $g \geq 2d - 2$ ),  $\gamma_2 = \frac{13}{9}$ , and  $\alpha_2 = \gamma_3 = \frac{99}{13}$ . Thus, if a graph with this spectrum exists, then it will not be distance-regular. Now if you consider the graph in the association scheme, then for both types of vertices at distance 2 from a fixed vertex (the type depending on  $c_2(x, y)$  being 1 or 2), you can count the number of walks of length 3 using the intersection numbers of the scheme, and indeed in both cases this number equals  $\alpha_2\gamma_2 = 11$ .

Here it is also worth noting that, under the conditions of the above proposition, the average number of walks  $\bar{a}_{d-1}^{(d)}$  coincides with the average of the product  $a_{d-1}(u, v)c_{d-1}(u, v)$  over all pairs  $(u, v)$  at distance  $d - 1$ . Indeed,

$$\begin{aligned}\bar{a}_{d-1}^{(d)} &= \frac{\langle A^d, A_{d-1} \rangle}{\|A_{d-1}\|^2} = \frac{\langle A^{d-1}, AA_{d-1} \rangle}{\|A_{d-1}\|^2} = \frac{1}{n\bar{k}_{d-1}} \text{sum}(A^{d-1} \circ AA_{d-1}) \\ &= \frac{1}{n\bar{k}_{d-1}} \sum_{u,v \in V} (A^{d-1})_{uv} (AA_{d-1})_{uv} = \frac{1}{n\bar{k}_{d-1}} \sum_{\text{dist}(u,v)=d-1} c_{d-1}(u, v) a_{d-1}(u, v),\end{aligned}$$

where we have used that  $(A^{d-1})_{uv} = 0$  when  $\text{dist}(u, v) \geq d$  and, since  $a_{d-2} = 0$ ,  $(AA_{d-1})_{uv} = 0$  when  $\text{dist}(u, v) \leq d - 2$ .

Note also that for a regular graph with girth  $g \geq 2d - 2$ , one can derive nice formulas for the predistance polynomials. Indeed, from  $\gamma_i = 1, \alpha_i = 0, \beta_i = k - 1$  for  $i = 0, \dots, d - 2$ , it follows by induction that

$$\gamma_i p_i = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i-j}{j} (1-k)^j x^{i-2j}$$

for  $i = 0, \dots, d - 1$ . From this, one can also get  $p_d$ , in particular that

$$\gamma_{d-1} \gamma_d p_d = x^d - \alpha_{d-1} x^{d-1} - (d-2 + \gamma_{d-1})(k-1) x^{d-2} + \alpha_{d-1}(d-2)(k-1) x^{d-3} + \dots$$

Then, by looking at the second term of the Hoffman polynomial (or  $p_d + p_{d-1}$ ), we obtain (6.4) again.

In order to give a generalization of Proposition 44 we now first need to recall the concepts of local multiplicity and walk-regular graph. For  $i = 0, \dots, d$ , let  $E_i$  be the (*minimal*) idempotent of  $A$  that corresponds to the orthogonal projection onto the eigenspace corresponding to  $\lambda_i$ . By analogy with the so-called *local multiplicities*, which correspond to the diagonal entries of the idempotents, Fiol, Garriga, and Yebra [30] defined the *crossed (uv-)local multiplicities* of the eigenvalue  $\lambda_i$ , denoted by  $m_{uv}(\lambda_i)$ , as

$$m_{uv}(\lambda_i) = (E_i)_{uv}, \quad u, v \in V; \ i = 0, \dots, d.$$

For regular graphs,  $E_0 = \frac{1}{n}J$  and, hence,  $m_{uv}(\lambda_0) = 1/n$  for every  $u, v \in V$ . The crossed local multiplicities allow us to express the number of walks of length  $\ell$  between two vertices  $u, v$  in the following way:

$$a_{uv}^{(\ell)} = (A^\ell)_{uv} = \sum_{i=0}^d m_{uv}(\lambda_i) \lambda_i^\ell, \quad \ell \geq 0.$$

A graph  $G$  with diameter  $D$  is called  *$h$ -punctually walk-regular*, for some  $h = 0, \dots, D$ , when for every  $\ell$ , the number of walks  $a_{uv}^{(\ell)}$  is the same for every pair of vertices  $u, v$  at distance  $h$ . From the above expression it follows that this is equivalent to the crossed local multiplicities  $m_{uv}(\lambda_i)$  being the same for every pair of vertices  $u, v$  at distance  $h$  (i.e., they only depend on  $i$ ; see Dalfó, Van Dam, Fiol, Garriga, and Gorissen [24] for more details). Moreover,  $G$  is called  *$m$ -walk-regular* for some  $m \leq D$  if it is  *$h$ -punctually walk-regular* for every  $h \leq m$  (see Dalfó, Fiol, and Garriga [22]).

As commented above, the following result is, in some aspects, a generalization of Proposition 44. In particular, note that the condition  $\lambda_1 + \dots + \lambda_d \neq 0$  below is equivalent, by Lemma 31(ii) and  $\alpha_d + \gamma_d = \lambda_0$ , to  $\gamma_d \neq \alpha_{d-1} + \dots + \alpha_0$ .

**Theorem 45.** *If  $G$  is a  $(d - 2)$ -partially distance-regular graph, the parameters  $a_{d-2}$  and  $a_{d-1}^{(d)}$  are well-defined, and  $\lambda_1 + \dots + \lambda_d \neq 0$ , then  $G$  is distance-regular.*

*Proof.* By considering the number of walks between two vertices  $u, v$  at dis-

tance  $d - 1$ , we obtain the following  $d$  equations:

$$\begin{aligned} \sum_{i=1}^d m_{uv}(\lambda_i) \lambda_i^\ell &= -m_{uv}(\lambda_0) \lambda_0^\ell = -\frac{1}{n} \lambda_0^\ell, & \ell = 0, \dots, d-2, \\ \sum_{i=1}^d m_{uv}(\lambda_i) \lambda_i^d &= -\frac{1}{n} \lambda_0^d + a_{d-1}^{(d)}. \end{aligned}$$

This can be seen as a determined system of  $d$  equations and  $d$  unknowns  $m_{uv}(\lambda_i)$ ,  $i = 1, \dots, d$ , since its coefficient matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_d \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{d-2} & \lambda_2^{d-2} & \cdots & \lambda_d^{d-2} \\ \lambda_1^d & \lambda_2^d & \cdots & \lambda_d^d \end{bmatrix}$$

is nonsingular. Indeed, if  $\sigma = \lambda_1 + \cdots + \lambda_d$ , then it follows from expanding the (Hoffman-like) polynomial  $\prod_{i \neq 0} (x - \lambda_i)$  that  $\lambda_i^d = \sigma \lambda_i^{d-1} + g_{d-2}(\lambda_i)$  for some polynomial  $g_{d-2}$  of degree at most  $d - 2$ , for all  $i \neq 0$ . Hence, the determinant of the coefficient matrix is  $\sigma$  times the determinant of the Vandermonde matrix  $\mathbf{V}$ , with entries  $(\mathbf{V})_{ij} = \lambda_i^{j-1}$  for all  $i, j = 1, \dots, d$ .

As a consequence, the crossed local multiplicities  $m_{uv}(\lambda_i)$  are the same for all vertices  $u, v$  at distance  $d - 1$  and  $G$  is  $(d - 1)$ -punctually walk-regular. In particular, the number of walks  $a_{uv}^{(d-1)} = a_{d-1}^{(d-1)}$  does not depend on the vertices  $u, v$  and, hence,

$$a_{d-1}^{(d-1)} = (A^{d-1})_{uv} = c_{d-1}(u, v).$$

So  $c_{d-1}$  is well-defined and, since  $a_{d-2}$  and  $b_{d-2} = \lambda_0 - c_{d-2} - a_{d-2}$  are also well-defined,  $G$  is  $(d - 1)$ -partially distance-regular, and the result follows again from Proposition 28(i).  $\square$

The above technique of computing the (crossed) local multiplicities through a system of equations has also been used to give a short proof of the odd-girth theorem (see Van Dam and Fiol [58]), and to prove that every pseudo-distance-regularized graph, which is a generalization of distance-regularized graphs in the sense of Godsil and Shawe-Taylor [34], is either distance-regular or distance-biregular (see Fiol [27]).

# An Interlacing Approach for Bounding the Sum of Laplacian Eigenvalues of Graphs

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In this chapter we apply eigenvalue interlacing techniques for obtaining lower and upper bounds for the sums of Laplacian eigenvalues of graphs, and characterize equality. This leads to generalizations of, and variations on theorems by Grone, and Grone and Merris. As a consequence we obtain inequalities involving bounds for some well-known parameters of a graph, such as edge-connectivity, and the isoperimetric number.

Recall that the Laplacian matrix of  $G$  is  $L = D - A$  where  $D$  is the diagonal matrix of the vertex degrees and  $A$  is the adjacency matrix of  $G$ .

First, for the sake of simplicity, we will recall the following basic result about interlacing (see [37], [25], or [14]).

**Theorem 46.** *Let  $A$  be a real symmetric  $n \times n$  matrix with eigenvalues  $\lambda_1 \geq \dots \geq \lambda_n$ . For some  $m < n$ , let  $S$  be a real  $n \times m$  matrix with orthonormal columns,  $S^\top S = I$ , and consider the matrix  $B = S^\top AS$ , with eigenvalues  $\mu_1 \geq \dots \geq \mu_m$ .*

(a) *The eigenvalues of  $B$  interlace those of  $A$ , that is,*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}, \quad i = 1, \dots, m, \quad (7.1)$$

- (b) If the interlacing is tight, that is, for some  $0 \leq k \leq m$ ,  $\lambda_i = \mu_i$ ,  $i = 1, \dots, k$ , and  $\mu_i = \lambda_{n-m+i}$ ,  $i = k+1, \dots, m$ , then  $SB = AS$ .

Eigenvalue interlacing will serve as a main tool in the proof of some results. As we saw in Chapter 2, Section 2.3, there are two cases of interlacing depending on how we choose the matrix  $S$ . If  $S = [I \ O]^\top$ , then  $B$  is a principal submatrix of  $A$ . If  $\mathcal{P} = \{U_1, \dots, U_m\}$  is a partition of  $\{1, \dots, n\}$  we can take for  $\tilde{B}$  the so-called quotient matrix of  $A$  with respect to  $\mathcal{P}$ .

The first case gives useful conditions for an induced subgraph  $G'$  of a graph  $G$ , because the adjacency matrix of  $G'$  is a principal submatrix of the adjacency matrix of  $G$ . However, the Laplacian matrix  $L'$  of  $G'$  is in general not a principal submatrix of the Laplacian matrix  $L$  of  $G$ . But  $L' + D'$  is a principal submatrix of  $L$  for some nonnegative diagonal matrix  $D'$ . Therefore the left hand inequalities in (7.1) still hold for the Laplacian eigenvalues, because adding the positive semidefinite matrix  $D'$  decreases no eigenvalue.

In the case that  $\tilde{B}$  is a quotient matrix of  $A$  with respect to  $\mathcal{P}$ , the element  $\tilde{b}_{ij}$  of  $\tilde{B}$  is the average row sum of the block  $A_{i,j}$  of  $A$  with rows and columns indexed by  $U_i$  and  $U_j$ , respectively. If  $\mathcal{P}$  has characteristic matrix  $C$  (that is, the columns of  $C$  are the characteristic vectors of  $U_1, \dots, U_m$ ) then we take  $S = CD^{-1/2}$ , where  $D = \text{diag}(|U_1|, \dots, |U_m|) = C^\top C$ . In this case, the quotient matrix  $\tilde{B}$  is in general not equal to  $B = S^\top AS$ , but  $\tilde{B} = D^{-1/2} S^\top A S D^{1/2}$ , and thus  $\tilde{B}$  is similar to (and therefore has the same spectrum as)  $B = S^\top AS$ .

If the interlacing is tight, then (b) of Theorem 46 reflects that  $\mathcal{P}$  is an equitable partition of  $A$ , that is, each block of the partition has constant row and column sums. In case  $A$  is the adjacency matrix of a graph  $G$ , equitability of  $\mathcal{P}$  implies that the bipartite induced subgraph  $G[U_i, U_j]$  is biregular for each  $i \neq j$ , and that the induced subgraph  $G[U_i]$  is regular for each  $i \in \{1, \dots, m\}$ . In case of tight interlacing for the quotient matrix of the Laplacian matrix of  $G$ , the first condition also holds, but the induced subgraphs  $G[U_i]$  are not necessarily regular (in this case we speak about an *almost equitable* partition of  $G$ ). In other words, given a symmetric matrix  $A$  with rows and columns indexed by a set  $V$  that is partitioned into  $m$  classes  $\{U_1, \dots, U_m\}$ , we say that the matrix partition is *equitable* whenever each block  $A_{i,j}$  has constant row (and column) sum. Similarly, a matrix partition is called *almost equitable* whenever each off-diagonal block  $A_{i,j}$  has constant row sum.

If a symmetric matrix  $A$  has an equitable partition, we have the following well-known and useful result ([14], Section 2.3).

**Lemma 47.** *Let  $A$  be a symmetric matrix of order  $n$ , and suppose  $\mathcal{P}$  is a partition of  $\{1, \dots, n\}$  such that the corresponding partition of  $A$  is equitable with quotient matrix  $\tilde{B}$ . Then the spectrum of  $B$  is a sub(multi)set of the spectrum of  $A$ , and all corresponding eigenvectors of  $A$  are in the column space of the characteristic matrix  $C$  of  $\mathcal{P}$  (this means that the entries of the eigenvector are constant on each partition class  $U_i$ ). The remaining eigenvectors of  $A$  are orthogonal to the columns of  $C$  and the corresponding eigenvalues remain unchanged if the blocks  $A_{i,j}$  are replaced by  $A_{i,j} + c_{i,j}J$  for certain constants  $c_{i,j}$ .*

The above lemma is a direct consequence of the fact that, if for an equitable partition  $\mathcal{P}$ ,  $\mathbf{v}$  is an eigenvector of  $\tilde{B}$  for an eigenvalue  $\lambda$ , then  $C\mathbf{v}$  is an eigenvector of  $A$  for the same eigenvalue  $\lambda$ , since  $\tilde{B}\mathbf{v} = \lambda\mathbf{v}$  implies  $AC\mathbf{v} = C\tilde{B}\mathbf{v} = \lambda C\mathbf{v}$ .

Assuming that  $G$  has  $n$  vertices, with degrees  $d_1 \geq d_2 \geq \dots \geq d_n$ , and Laplacian matrix  $L$  with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (= 0)$ , it is known that, for  $1 \leq m \leq n$ ,

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m d_i. \quad (7.2)$$

This is a consequence of Schur's theorem [55] stating that the spectrum of any symmetric, positive semidefinite matrix majorizes its main diagonal. In particular, note that if  $m = n$  we have equality in (7.2), because both terms correspond to the trace of  $L$ . To prove (7.2) by using interlacing, let  $B$  be a principal  $m \times m$  submatrix of  $L$  indexed by the subindices corresponding to the  $m$  higher degrees, with eigenvalues  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ . Then,

$$\text{tr } B = \sum_{i=1}^m d_i = \sum_{i=1}^m \mu_i,$$

and, by interlacing,  $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$  for  $i = 1, \dots, m$ , whence (7.2) follows. Similarly, reasoning with the principal submatrix  $B$  (of  $L$ ) indexed by the  $m$  vertices with lower degrees we get:

$$\sum_{i=1}^m \lambda_{n-m+i} \leq \sum_{i=1}^m d_{n-m+i}. \quad (7.3)$$

The next result, which is an improvement of (7.2), is due to Grone [35], who



proved that if  $G$  is connected and  $m < n$  then,

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m d_i + 1. \quad (7.4)$$

In [14], Brouwer and Haemers gave two different proofs of (7.4), both using eigenvalue interlacing. In this chapter we extend the ideas of these two proofs and find a generalization of Grone's result (7.4), and another lower bound on the sum of the largest Laplacian eigenvalues, which is closely related to a bound of Grone and Merris [36].

## 7.1 A generalization of Grone's result

Throughout this chapter, we will denote our “quotient” matrix  $B$  instead of  $\tilde{B}$ , since we will make use of both types of interlacing: when  $B$  is the principal submatrix of  $A$  and when  $\tilde{B}$  is the quotient matrix of  $A$  with respect to a certain partition  $\mathcal{P}$ .

We begin with a basic result from where most of our bounds derive. Given a graph  $G$  with a vertex subset  $U \subset V$ , let  $\partial U$  be the *vertex-boundary* of  $U$ , that is, the set of vertices in  $\bar{U} = V \setminus U$  with at least one adjacent vertex in  $U$ . Also, let  $\partial(U, \bar{U})$  denote the *edge-boundary* of  $U$ , which is the set of edges which connect vertices in  $U$  with vertices in  $\bar{U}$ .

**Theorem 48.** *Let  $G$  be a connected graph on  $n = |V|$  vertices, having Laplacian matrix  $L$  with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (= 0)$ . For any given vertex subset  $U = \{u_1, \dots, u_m\}$  with  $0 < m < n$ , we have*

$$\sum_{i=1}^m \lambda_{n-i} \leq \sum_{u \in U} d_u + \frac{|\partial(U, \bar{U})|}{n-m} \leq \sum_{i=1}^m \lambda_i. \quad (7.5)$$

*Proof.* Consider the partition of the vertex set  $V$  into  $m+1$  parts:  $U_i = \{u_i\}$  for  $u_i \in U$ ,  $i = 1, \dots, m$ , and  $U_{m+1} = \bar{U}$ . Then, the corresponding quotient matrix is

$$B = \left[ \begin{array}{c|c} L_U & \begin{matrix} b_{1,m+1} \\ \vdots \\ b_{m,m+1} \end{matrix} \\ \hline \begin{matrix} b_{m+1,1} & \cdots & b_{m+1,m} \end{matrix} & b_{m+1,m+1} \end{array} \right],$$

where  $L_U$  is the principal submatrix of  $L$ , with rows and columns indexed by the vertices in  $U$ ,  $b_{i,m+1} = (n-m)b_{m+1,i} = -|\partial(u_i, \bar{U})|$ , and  $b_{m+1,m+1} = |\partial(U, \bar{U})|/(n-m)$  (because  $\sum_{i=1}^{m+1} b_{m+1,i} = 0$ ). Let  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{m+1}$  be the eigenvalues of  $B$ . Since  $B$  has row sum 0, we have  $\mu_{m+1} = \lambda_n = 0$ . Moreover,

$$\sum_{i=1}^m \mu_i = \sum_{i=1}^{m+1} \mu_i = \text{tr } B = \sum_{u \in U} d_u + b_{m+1,m+1},$$

Then, (7.5) follows by applying interlacing,  $\lambda_i \geq \mu_i \geq \lambda_{n-m-1+i}$  and adding up for  $i = 1, 2, \dots, m$ .  $\square$

If equality holds on either side of (7.5) it follows that the interlacing is tight (see the proof of Proposition 49 for details), and therefore that the partition of  $G$  is almost equitable. In other words, in case of equality every vertex  $u \in U$  is adjacent to either all or 0 vertices in  $\bar{U}$ , whereas each vertex  $u \in \bar{U}$  has precisely  $|\partial(U, \bar{U})|/(n-m)$  neighbors in  $U$ . But we can be more precise.

**Proposition 49.** *Let  $H$  be the subgraph of  $G$  induced by  $\bar{U}$ , and let  $\vartheta_1 \geq \dots \geq \vartheta_{n-m} (= 0)$  be the Laplacian eigenvalues of  $H$ . Define  $b = |\partial(U, \bar{U})|/(n-m)$ .*

- (a) *Equality holds on the right hand side of (7.5) if and only if each vertex of  $U$  is adjacent to all or 0 vertices of  $\bar{U}$ , and  $\lambda_{m+1} = \vartheta_1 + b$ .*
- (b) *Equality holds on the left hand side of (7.5) if and only if each vertex of  $U$  is adjacent to all or 0 vertices of  $\bar{U}$ , and  $\lambda_{n-m-1} = \vartheta_{n-m} + b$ .*

*Proof.* Since (a) and (b) have analogous proofs, we only prove (a). Suppose equality holds on the right hand side of (7.5). Then

$$\sum_{i=1}^m \lambda_i = \sum_{i=1}^m \mu_i, \text{ and } \lambda_i \geq \mu_i \text{ for } i = 1, \dots, m$$

so  $\lambda_i = \mu_i$  for  $i = 1, \dots, m$ . We know that  $\mu_{m+1} = \lambda_n = 0$ , therefore the interlacing is tight and hence the partition of  $G$  is almost equitable. Now by use of Lemma 47 we have that the eigenvalues of  $L$  are  $\mu_1, \dots, \mu_{m+1}$  together with the eigenvalues of  $L$  with an eigenvector orthogonal to the characteristic matrix  $C$  of the partition. These eigenvalues and eigenvectors remain unchanged if  $L$  is changed into

$$\tilde{L} = \begin{bmatrix} O & O \\ O & L_{\bar{U}} + bI \end{bmatrix}.$$

The considered common eigenvalues of  $\tilde{L}$  and  $L$  are  $\vartheta_1 + b \geq \cdots \geq \vartheta_{n-m-1} + b$ . So  $L$  has eigenvalues  $\lambda_1(= \mu_1) \geq \cdots \geq \lambda_m(= \mu_m)$ , and  $\vartheta_1 + b \geq \cdots \geq \vartheta_{n-m-1} + b \geq \lambda_n(= \mu_{m+1} = 0)$ . Hence, we have  $\lambda_{m+1} = \vartheta_1 + b$ . Conversely, if the partition of  $G$  is almost equitable (or equivalently, if the partition of  $L$  is equitable),  $L$  has eigenvalues  $\mu_1 \geq \cdots \geq \mu_m$ ,  $\vartheta_1 + b \geq \cdots \geq \vartheta_{n-m-1} + b$ , and  $\mu_{m+1} = \lambda_n = 0$ . Since  $\lambda_{m+1} = \vartheta_1 + b$ , it follows that  $\mu_i = \lambda_i$  for  $i = 1, \dots, m$  (tight interlacing), therefore equality holds on the right hand side of (7.5).  $\square$

Looking for examples of the above results, first observe that there is no graph with  $n > 2$  satisfying equality in (7.4) for every  $0 < m < n$ . However the complete graph  $K_n$  provides an example for which both inequalities in Theorem 48 are equalities for all  $0 < m < n$ . In fact, this is a particular case of the following construction (just take  $q = 1$ ): Let us consider the *graph join*  $G$  of the complete graph  $K_p$  with the empty graph  $\overline{K}_q$ . (Recall that  $G$  is obtained as the graph union of  $K_p$  and  $\overline{K}_q$  with all the edges connecting the vertices of one graph with the vertices of the other.) Let  $V(G) = \{v_1, \dots, v_p, v_{p+1}, \dots, v_n\}$ , where  $n = p + q$  and the first vertices correspond to those of  $K_p$ . Then, the Laplacian eigenvalues of  $G$  are  $\{n^p, p^{q-1}, 0^1\}$ , and the following different choices for  $U$  provide some examples illustrating cases (a) and (b) of Proposition 49.

(a1) Let  $U = \{v_1, \dots, v_m\}$ , with  $0 < m \leq p$ . Then,  $b = m$ , and

$$\sum_{u \in U} d_u + b = m(n-1) + m = mn = \sum_{i=1}^m \lambda_i.$$

(a2) Let  $U = \{v_1, \dots, v_m\}$ , with  $p < m < n$ . Then,  $b = p$ , and

$$\sum_{u \in U} d_u + b = p(n-1) + (m-p)p + p = pn + (m-p)p = \sum_{i=1}^m \lambda_i.$$

(b) Let  $U = \{v_{n-m+1}, \dots, v_n\}$ , with  $q \leq m < n$ . Then,  $b = m$ , and

$$\sum_{u \in U} d_u + b = qp + (m-q)(n-1) + m = (q-1)p + (m-q+1)n = \sum_{i=1}^m \lambda_{n-i}.$$

Another infinite family of graphs for which we do have equality on the right hand side of (7.5) is the complete multipartite graph (such that the vertices with largest degree lie in  $U$ ).

If the vertex degrees of  $G$  are  $d_1 \geq d_2 \geq \dots \geq d_n$ , we can choose conveniently the  $m$  vertices of  $U$  (that is, those with higher or lower degrees) to obtain the best inequalities in (7.5). Namely,

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m d_i + \frac{|\partial(U, \bar{U})|}{n-m}, \quad (7.6)$$

and

$$\sum_{i=1}^m \lambda_{n-i} \leq \sum_{i=1}^m d_{n-i+1} + \frac{|\partial(U, \bar{U})|}{n-m}. \quad (7.7)$$

Note that (7.7), together with (7.3) for  $m+1$ , yields

$$\sum_{i=0}^m \lambda_{n-m+i} = \sum_{i=1}^m \lambda_{n-i} \leq \sum_{i=1}^m d_{n-i+1} + \min \left\{ d_{n-m}, \frac{|\partial(U, \bar{U})|}{n-m} \right\}. \quad (7.8)$$

If we have more information on the structure of the graph, we can improve the above results by either bounding  $|\partial(U, \bar{U})|$  or ‘optimizing’ the ratio  $b = |\partial(U, \bar{U})|/(n-m)$ . In fact, the right inequality in (7.5) (and, hence, (7.6)) can be improved when  $\bar{U} \neq \partial U$ . Simply first delete the vertices (and corresponding edges) of  $\bar{U} \setminus \partial U$ , and then apply the inequality. Then  $d_1, \dots, d_m$  remain the same and  $\lambda_1, \dots, \lambda_m$  do not increase (see Lemma 2.3 in [48]). Thus we obtain:

**Theorem 50.** *Let  $G$  be a connected graph on  $n = |V|$  vertices, with Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (= 0)$ . For any given vertex subset  $U = \{u_1, \dots, u_m\}$  with  $0 < m < n$ , we have*

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in U} d_u + \frac{|\partial(U, \bar{U})|}{|\partial U|}. \quad (7.9)$$

Similarly as we did in (7.6), if we choose the  $m$  vertices of  $U$  such that they are those with maximum degree, then we can write:

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m d_i + \frac{|\partial(U, \bar{U})|}{|\partial U|}.$$

Notice that, as a corollary, we get Grone's result (7.4) since always  $|\partial(U, \bar{U})| \geq |\partial U|$ .

## 7.2 A variation of a bound by Grone and Merris

In [36], Grone and Merris gave another lower bound for the sum of the Laplacian eigenvalues, in the case when there is an induced subgraph consisting of isolated vertices and edges. Let  $G$  be a connected graph of order  $n > 2$  with Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . If the induced subgraph of a subset  $U \subset V$  with  $|U| = m$  consists of  $r$  pairwise disjoint edges and  $m - 2r$  isolated vertices, then

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in U} d_u + m - r. \quad (7.10)$$

An improvement of this result was given by Brouwer and Haemers in [14] (Section 3.10). Let  $G$  be a (not necessarily connected) graph with a vertex subset  $U$ , with  $m = |U|$ , and let  $h$  be the number of connected components of  $G[U]$  that are not connected components of  $G$ . Then,

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in U} d_u + h. \quad (7.11)$$

Following the same ideas as in [36] and using interlacing, the bound (7.10) of Grone and Merris can also be generalized as follows:

**Theorem 51.** *Let  $G$  be a connected graph of order  $n > 2$  with Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Given a vertex subset  $U \subset V$ , with  $m = |U| < n$ , let  $G[U] = (U, E[U])$  be its induced subgraph. Then,*

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in U} d_u + m - |E[U]|. \quad (7.12)$$

*Proof.* Consider an orientation of  $G$  with all edges in  $E(U, \bar{U})$  oriented from  $U$  to  $\bar{U}$ , and every vertex in  $U \setminus \partial \bar{U}$  having some outgoing arc (this is always possible as  $G$  is connected). Let  $Q$  be the corresponding oriented incidence matrix of  $G$ , and write  $Q = [Q_1 \ Q_2]$ , where  $Q_1$  corresponds to  $E[U] \cup E(U, \bar{U})$ , and  $Q_2$  corresponds to  $E[\bar{U}]$ . Consider the matrix  $M = Q^\top Q$ , with entries  $(M)_{ii} = 2$ ,  $(M)_{ij} = \pm 1$  if the arcs  $e_i, e_j$  are incident to the same vertex (+1 if both are either outgoing or ingoing, and  $-1$  otherwise), and  $(M)_{ij} = 0$  if the corresponding edges are disjoint, and define  $M_1 = Q_1^\top Q_1$ . Then  $M$  has the same nonzero eigenvalues as  $L = QQ^\top$ , the Laplacian

matrix of  $G$ , and  $M_1$  is a principal submatrix of  $M$ . For every vertex  $u \in U$ , let  $E_u$  be the set of outgoing arcs from  $u$ . Then  $\{E_u \mid u \in U\}$  is a partition of  $E[U] \cup E(U, \bar{U})$ . Consider the quotient matrix  $B_1 = (b_{ij})$  of  $M_1$  with respect to this partition. Then,  $b_{uu} = d^+(u) + 1$  for each  $u \in U$ . Let  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$  be the eigenvalues of  $B_1$ , then

$$\text{tr } B_1 = \sum_{i=1}^m \mu_i = \sum_{u \in U} d_u^+ + m = \sum_{u \in U} d_u - |E[U]| + m$$

and (7.12) follows since the eigenvalues of  $B_1$  interlace those of  $M_1$ , which in turn interlace those of  $M$ .  $\square$

Note that (7.12) also follows from Equation (7.11). However, the result can be improved by considering the partition  $\mathcal{P} = \{E_u \mid u \in U\} \cup \{E[\bar{U}]\}$  of the whole edge set of  $G$ .

**Theorem 52.** *Let  $G$  be a connected graph of order  $n > 2$  with Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Given a vertex subset  $U \subset V$ , with  $m = |U| < n$ , let  $G[U] = (U, E[U])$  and  $G[\bar{U}]$  be the corresponding induced subgraphs. Let  $\vartheta_1$  be the largest Laplacian eigenvalue of  $G[\bar{U}]$ , then*

$$\sum_{i=1}^{m+1} \lambda_i \geq \sum_{u \in U} d_u + m - |E[U]| + \vartheta_1. \quad (7.13)$$

*Proof.* First observe that the Laplacian matrix of  $G[\bar{U}]$  is  $Q_2 Q_2^\top$ , and therefore  $\vartheta_1$  is also the largest eigenvalue of  $Q_2^\top Q_2$ . Next we apply interlacing to an  $(m+1) \times (m+1)$  quotient matrix  $B = S^\top M S$ , which is defined slightly different as before. The first  $m$  columns of  $S$  are the normalized characteristic vectors of  $E_u$  (as before), but the last column of  $S$  equals  $\begin{bmatrix} \mathbf{0} \\ \mathbf{v} \end{bmatrix}$ , where  $\mathbf{v}$  is a normalized eigenvector of  $Q_2^\top Q_2$  for the eigenvalue  $\vartheta_1$ . Then  $b_{m+1, m+1} = \mathbf{v}^\top Q_2^\top Q_2 \mathbf{v} = \vartheta_1$ , and we find  $\text{tr } B = \sum_{u \in U} d_u + m - |E[U]| + \vartheta_1$ .  $\square$

### 7.3 Some applications

The previous bounds on the sum of Laplacian eigenvalues are used to provide meaningful results involving the edge-connectivity of the graph, the size of a  $k$ -dominating set and the isoperimetric number.

### 7.3.1 Cuts

Given a vertex subset  $U$  of a connected graph  $G$  with  $0 < |U| < n$ , the edge set  $\partial(U, \bar{U})$  is called a *cut* (since deletion of these edges makes  $G$  disconnected). The minimum size of a cut in  $G$  is called the *edge-connectivity*  $\kappa_e(G)$  of  $G$ . By use of inequality (7.6) we obtain the following bound for  $\kappa_e(G)$ .

**Proposition 53.**

$$\kappa_e(G) \leq \min_{0 < m < n} \left\{ (n - m) \sum_{i=1}^m (\lambda_i - d_i) \right\}. \quad (7.14)$$

Some general bounds on the size of a cut can be derived from the following lemma.

**Lemma 54.** *Let  $G$  be a graph with  $n$  vertices and  $e$  edges. For any  $m$ ,  $0 < m < n$ , there exist some (not necessarily different) vertex subsets  $U$  and  $U'$  such that  $|U| = |U'| = m$  and*

$$|\partial(U, \bar{U})| \geq \frac{2em(n - m)}{n(n - 1)}, \quad |\partial(U', \bar{U}')| \leq \frac{2em(n - m)}{n(n - 1)}. \quad (7.15)$$

*Proof.* Choose a set  $S$  uniformly at random among all the sets of size  $m$  in  $V$ . Then the probability that an edge belong to  $\partial(S, \bar{S})$  is the probability that either the first endpoint belongs to  $S$  and the second one to  $\bar{S}$  or viceversa. That is,

$$\Pr(\text{edge} \in \partial(S, \bar{S})) = 2 \frac{m(n - m)}{n(n - 1)}.$$

Then, the expected number of edges between the two sets is,

$$\mathbb{E}\{|\partial(S, \bar{S})|\} = \frac{2em(n - m)}{n(n - 1)},$$

implying that there are sets,  $U$  and  $U'$ , with at least and at most this number of edges going out, respectively.  $\square$

Both bounds are tight for the complete graph  $K_n$ . Using bounds (7.15), Theorem 48 gives:

**Corollary 55.** *For each  $m$  ( $0 < m < n$ )  $G$  has vertex sets  $U$  and  $U'$  of size  $m$  such that*

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in U} d_u + \frac{2em}{n(n-1)}, \quad (7.16)$$

and

$$\sum_{i=1}^m \lambda_{n-i} \leq \sum_{u \in U'} d_u + \frac{2em}{n(n-1)}. \quad (7.17)$$

In particular, if  $G$  is  $d$ -regular, we have  $e = nd/2$  and the above inequalities become

$$\sum_{i=1}^m \lambda_i \geq \frac{mdn}{n-1} \quad \text{and} \quad \sum_{i=1}^m \lambda_{n-i} \leq \frac{mdn}{n-1}, \quad (7.18)$$

with bounds close to  $md$  when  $n$  is large.

If  $G$  is  $d$ -regular we can look at the bounds in 7.18 in terms of the adjacency matrix  $A$  of  $G$ . Suppose that  $A$  has eigenvalues  $\theta_1 \leq \theta_2 \leq \dots \leq \theta_n = d$ , then the left inequality of 7.18 becomes

$$-\sum_{i=1}^m \theta_i \geq \frac{md}{n-1}, \quad (7.19)$$

which is clearly tight for the complete graph  $K_n$ , for example. As expected, if  $m = n - 1$ , then we have equality in 7.19 (indeed, it follows from  $\theta_n = d$  and  $\text{Tr}(A) = \sum_{i=1}^n \theta_i = 0$ ). Under the same assumptions, Schur's bound 7.2 becomes

$$-\sum_{i=1}^m \theta_i \geq 0,$$

and Grone's bound (7.4) gives

$$-\sum_{i=1}^m \theta_i \geq 1.$$

### 7.3.2 $k$ -Dominating sets

A *dominating set* in a graph  $G$  is a vertex subset  $D \subseteq V$  such that every vertex in  $V \setminus D$  is adjacent to some vertex in  $D$ . More generally, for a given integer  $k$ , a  *$k$ -dominating set* in a graph  $G$  is a vertex subset  $D \subseteq V$  such that every vertex in  $V \setminus D$  has at least  $k$  neighbors in  $D$ .



**Proposition 56.** *Let  $G$  be a graph on  $n$  vertices, with vertex degrees  $d_1 \geq d_2 \geq \dots \geq d_n$ , and Laplacian eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n (= 0)$ . Let  $D$  be a  $k$ -dominating set in  $G$  of cardinality  $m$ . Then,*

$$\sum_{i=1}^m \lambda_i \geq \sum_{u \in D} d_u + k. \quad (7.20)$$

*Proof.* First, the inequality (7.20) follows from Theorem 48 by noting that, from the definition of a  $k$ -dominating set,  $|\partial(D, \overline{D})| \geq k(n - m)$ .  $\square$

**Example 57.** Consider the  $K_{p, \dots, p}$  regular complete multipartite graph with  $q$  classes of size  $p$ , so  $n = pq$  and  $d = p(q-1)$ . The eigenvalues of its Laplacian matrix are

$$\{(d+p)^{q-1}, d^{n-q}, 0^1\}.$$

Observe that the union of some partition classes gives a  $k$ -dominating set of size  $m = k$ . If we take the first  $k$  eigenvalues, the inequality (7.20) becomes  $(d+p)(q-1) + (k - (q-1))d \geq kd + k$ , and using that  $d = p(q-1)$  we get  $d(k+1) \geq k(d+1)$ . Note that if  $k = d$  we have equality.

### 7.3.3 The isoperimetric number

Given a graph  $G$  on  $n$  vertices, the *isoperimetric number*  $i(G)$  is defined as

$$i(G) = \min_{U \subset V} \{|\partial(U, \overline{U})|/|U| : 0 < |U| \leq n/2\}.$$

For example, the isoperimetric numbers of the complete graph, the path and the cycle are, respectively,  $i(K_n) = \lceil \frac{n}{2} \rceil$ ,  $i(P_n) = 1/\lfloor \frac{n}{2} \rfloor$ , and  $i(C_n) = 2/\lfloor \frac{n}{2} \rfloor$ . For general graphs, Mohar [50] proved the following spectral bounds.

$$\frac{\lambda_{n-1}}{2} \leq i(G) \leq \sqrt{\lambda_{n-1}(2d_1 - \lambda_{n-1})}. \quad (7.21)$$

In our context we have:

**Proposition 58.**

$$i(G) \leq \min_{\frac{n}{2} \leq m < n} \sum_{i=1}^m (\lambda_i - d_i). \quad (7.22)$$

*Proof.* Apply (7.6) taking into account that  $i(G) \leq \frac{|\partial(\overline{U}, U)|}{|\overline{U}|}$  when  $0 < |\overline{U}| \leq \frac{n}{2}$ .  $\square$

**Example 59.** Consider the graph join  $G$  of the complete graph  $K_p$  with the empty graph  $\overline{K_q}$ , so  $n = p + q$ . The Laplacian spectrum and the degree sequence are

$$\{n^p, p^{q-1}, 0^1\} \quad \text{and} \quad \{(n-1)^p, p^q\},$$

respectively. Equation (7.22) gives  $i(G) \leq \min\{p, \lceil \frac{n}{2} \rceil\}$ , which is better than Mohar's upper bound (7.21) for all  $0 \leq q < n$ .



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